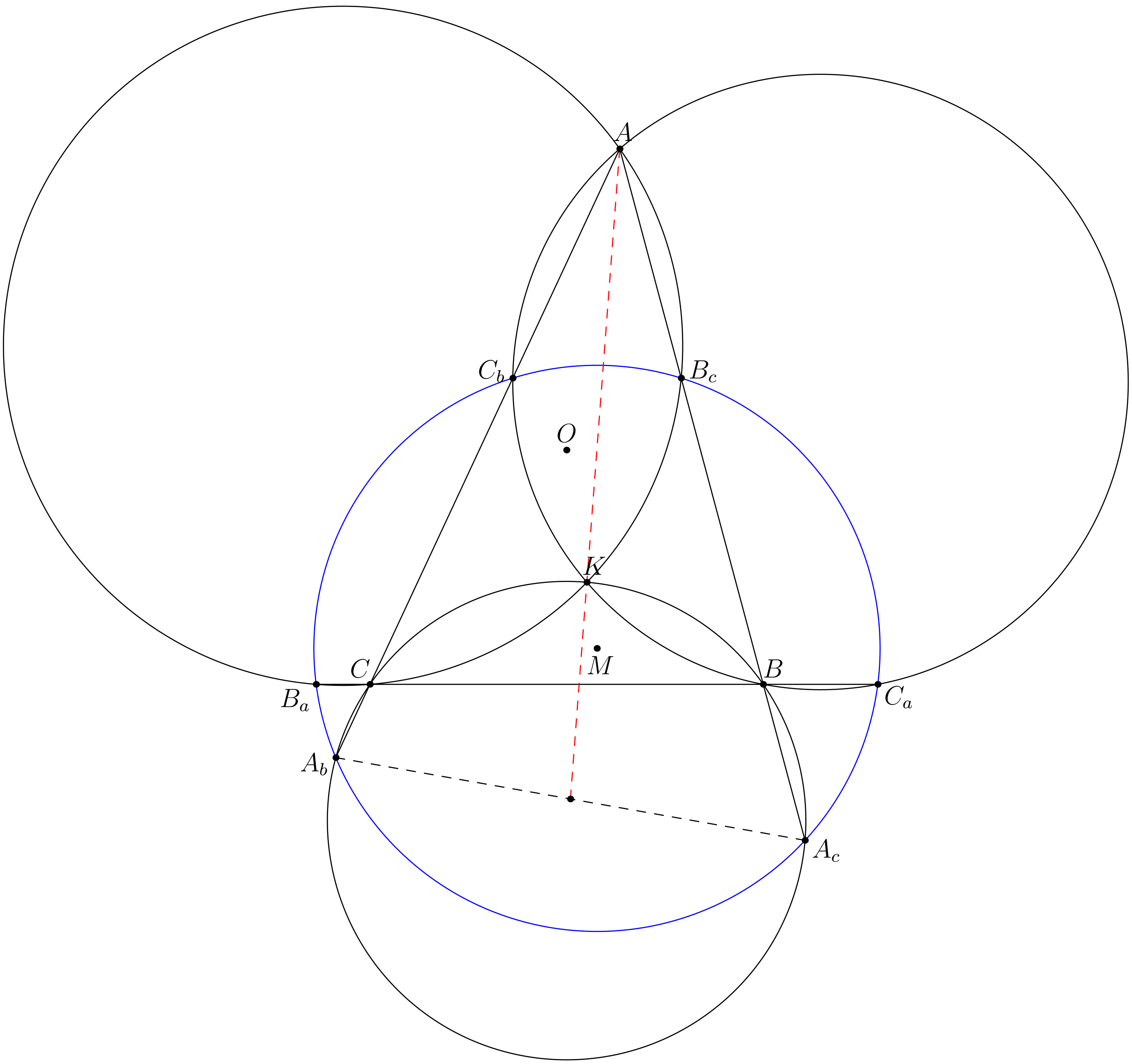
Problem 1: Ehrmann's Third Lemoine Circle [2]

Let ABC be a triangle with circumcenter O and Lemoine point K. The circumcircle of KBC intersects lines AB and AC again at points A_c and A_b, respectively. B_a, B_c, C_b, C_a are defined similarly. Prove that $A_cA_bB_aB_cC_bC_a$ is cyclic and that its circumcenter M lies on line OK such that $OK : KM = 2 : 1$.

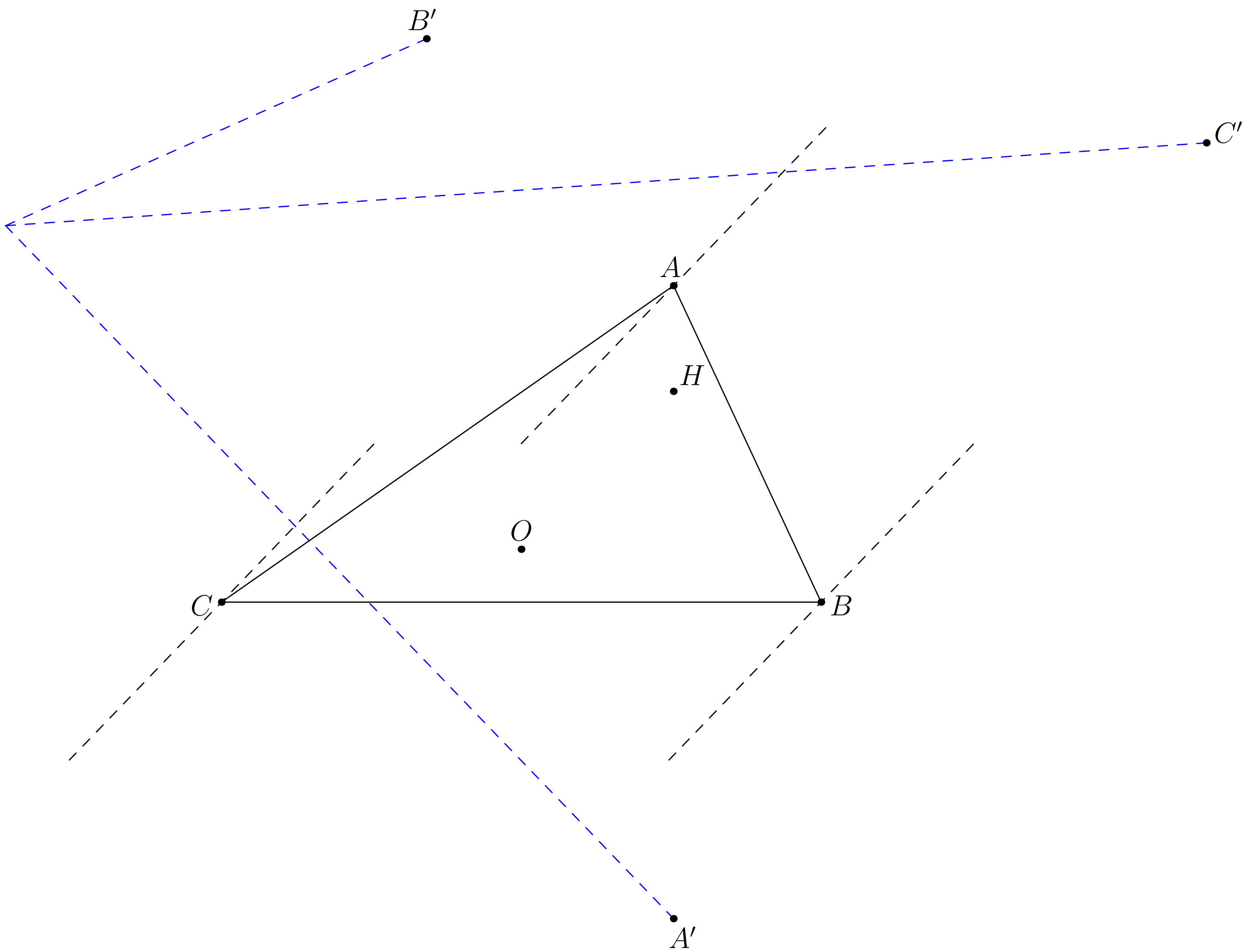


a) Prove that K is the centroid of $\triangle AA_cA_b$.

b) Prove that $B_cC_b || BC$.

Problem 2: Parry Reflection Point [3]

Let ABC be a triangle and let α, β, γ be three parallel lines passing through A, B, and C, respectively. Let α' be the reflection of α over BC and define β' and γ' similarly. Prove that α', β', and γ' concur if and only if α, β, γ are parallel to the Euler line of ABC.

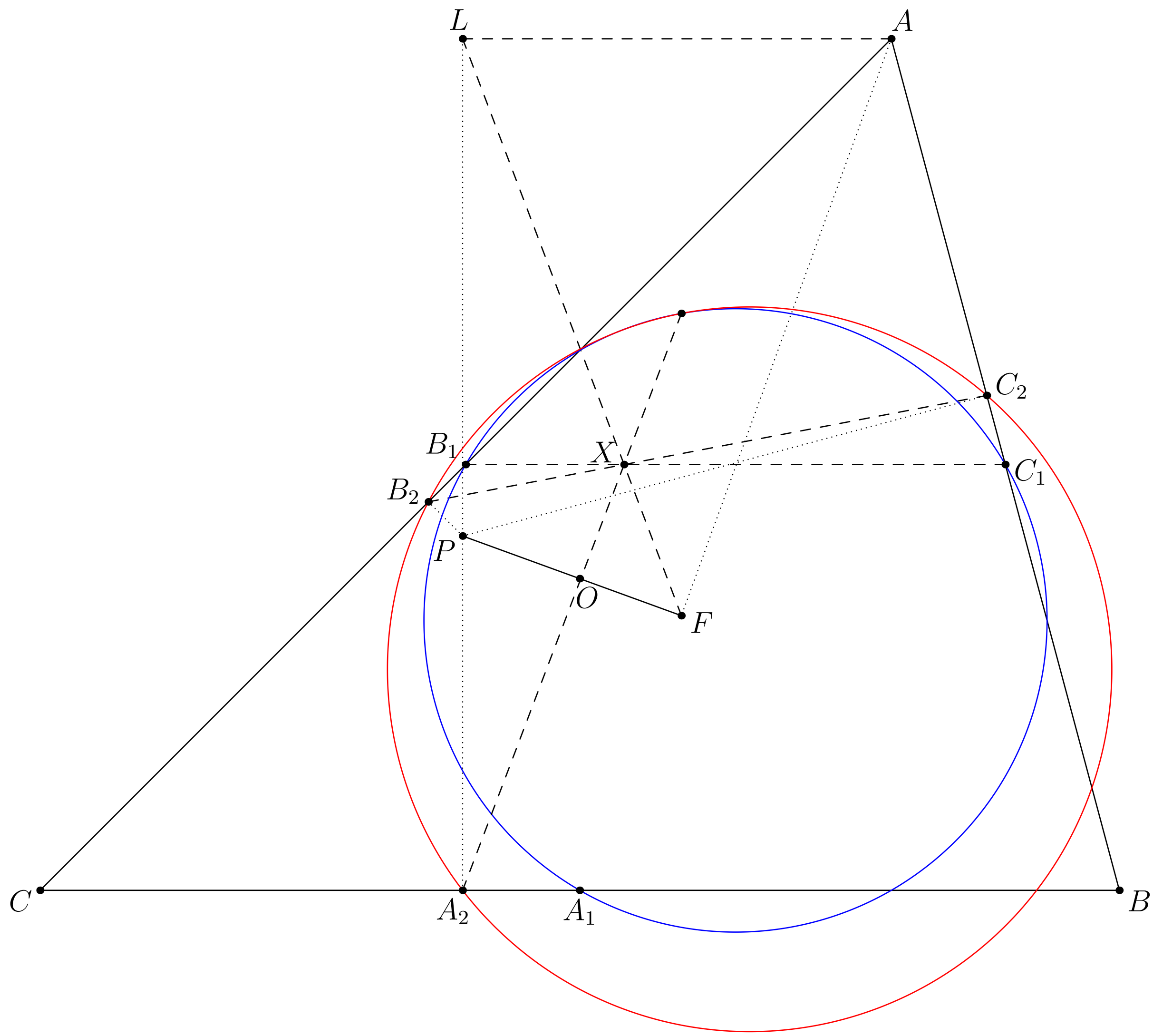


a) (Anti-Steiner Point). Let ℓ be a line in the plane of a triangle ABC. Prove that its reflections in the sidelines BC, CA, and AB are concurrent if and only if ℓ passes through the orthocenter H of ABC. In this case, their point of concurrency lies on the circumcircle.

b) Let P be a point in the plane of ABC and let ℓ be a line parallel to α, β, γ and passing through P. Prove that the bisectors of the three angles formed by ℓ with each of α', β', and γ' form a triangle homothetic to ABC.

Problem 3: Fontene's Third Theorem [4]

Let ABC be a triangle with circumcenter O. Let P and Q be isogonal conjugates with respect to triangle ABC. Prove that the pedal circle of P is tangent to the nine-point circle of ABC if and only if O, P, and Q are collinear.



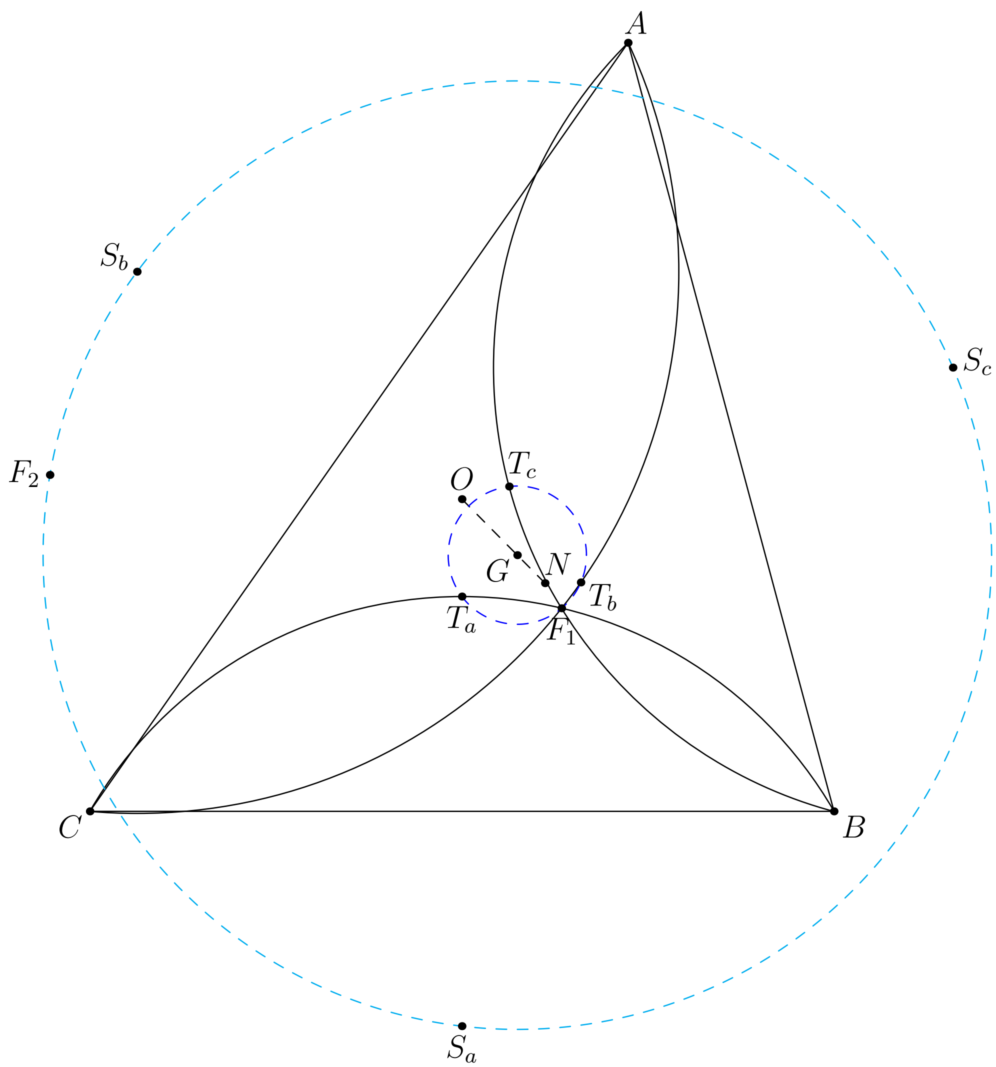
a) Let ABC be a triangle with circumcenter O and let P be a point in the plane. Let $A_1B_1C_1$ be the medial triangle of ABC and let $A_2B_2C_2$ be the pedal triangle of P with respect to triangle ABC. Let L be the reflection of A_2 over line B_1C_1. Let F be the foot of the perpendicular from A to line OP. Prove that FL, B_1C_1, and B_2C_2 are concurrent.

b) (Fontene's First Theorem). With the same points above, let X be the intersection of B_1C_1 and B_2C_2. Define Y and Z similarly. Prove that A_2X, B_2Y, C_2Z are concurrent, and that the point of concurrency lies on the circumcircle of $A_1B_1C_1$ and the circumcircle of $A_2B_2C_2$.

c) (Fontene's Second Theorem). If a point P moves on a fixed line ℓ which passes through the circumcenter O of ABC, then the pedal circle of P intersects the nine-point circle of ABC at a fixed point.

Problem 4: Lester's Theorem [1]

Let ABC be a triangle with circumcenter O, nine-point center N, and Fermat points F_1 and F_2. Prove that O, N, F_1, F_2 are concyclic.



a) (Fermat Points.) Equilateral triangles BCA_1 and BCA_2 are drawn such that A_1, A are on opposite sides of BC and A_2, A are on the same side of BC. B_1, B_2, C_1, C_2 are constructed similarly. Prove that the circumcircles of BCA_1, CAB_1, ABC_1 concur at a point F_1 and the circumcircles of BCA_2, CAB_2, ABC_2 concur at a point F_2. These two points of concurrency are known as the first and second Fermat points, respectively.

b) Let G be the centroid of ABC. Let T_a and S_a be the circumcenters of BCA_1 and BCA_2, respectively and define T_b, S_b, T_c, S_c similarly. Prove that $S_aS_bS_cF_1$ and $T_aT_bT_cF_2$ are two cyclic quadrilaterals with circumcenter G.

c) Let XYZ be a triangle and let Y' and Z' be the reflections of Y and Z over XZ and XY, respectively. Let ℓ be the tangent to the circumcircle of $XY'Z'$ at X. Lines YZ and $Y'Z'$ intersect ℓ at points W and W' respectively. Prove that X is the midpoint of WW'.

d) Prove that the Euler line of ABC is tangent to the circumcircle of GF_1F_2.

References

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