A Few Configurations

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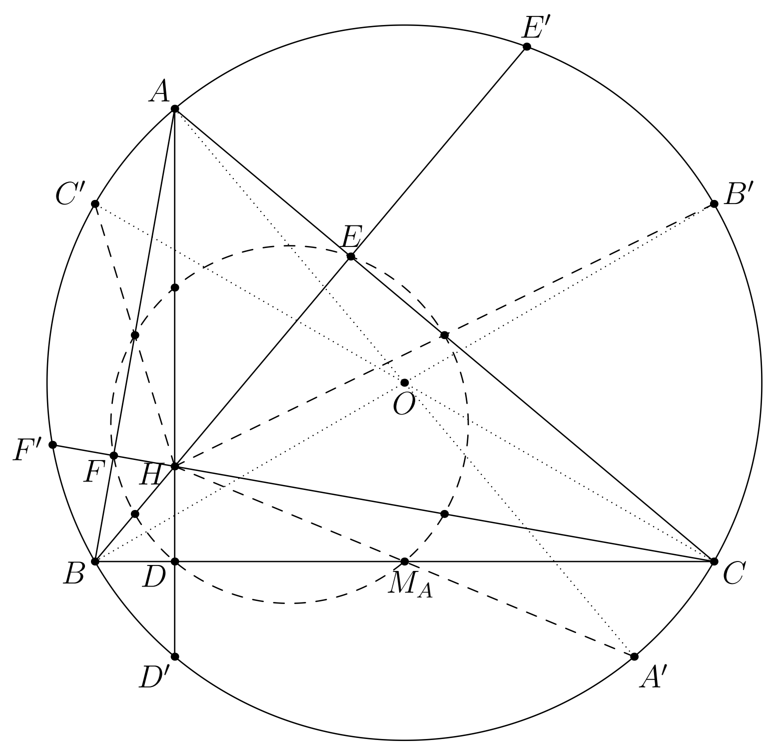
1 Around the Orthocenter



Let $\triangle ABC$ have orthocenter H and circumcenter O. Denote the circumcircle of $\triangle ABC$ by Γ. Let D, E, and F be the feet of the altitudes and M_A, M_B, and M_C be the midpoints. Let D', E', and F' be the second intersections of AH, BH, and CH respectively with Γ. Let A', B', and C' be the antipodes of A, B, and C with respect to Γ.

Fact 1.1. $\angle BHC = \pi - \angle A$.

Fact 1.2. Any one of A, B, C, H is the orthocenter of the triangle formed by the other three.

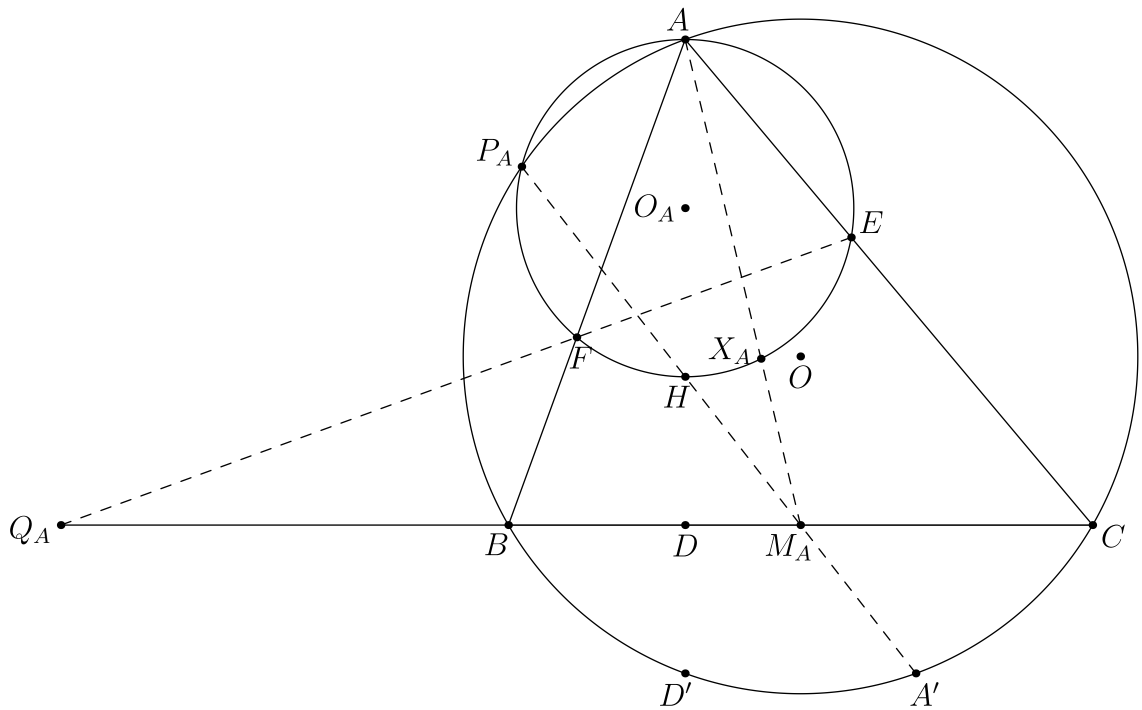
Fact 1.3. D' is the reflection of H over BC.

Fact 1.4. $D'A' \parallel BC$.

Fact 1.5. M_A is the midpoint of HA'.

Lemma 1 (Nine-Point Circle)**.** D, E, F, M_A, M_B, M_C lie on a common circle. Furthermore, the midpoints of AH, BH, and CH lie on this circle. The center of this circle is the midpoint of OH.

Proof. From the previous facts, D is the midpoint of HD' and M_A is the midpoint of HA'. Then a homothety centered about H with scale factor $\frac{1}{2}$ sends D' to D and A' to M_A. Similarly, B', E', C', and F' are sent to M_B, E, M_C, and F respectively. These points lie on Γ, so the mapped points lie on the scaled-down circle, known as the nine-point circle. The rest of the lemma follows easily from the homothety. $\square$



Let O_A be the midpoint of AH. Let P_A be the second intersection of $A'H$ and Γ. Let Q_A be the intersection of EF and BC. Let X_A be the foot of H on line AM_A. X_A (along with X_B and X_C defined similarly) is known as an HM-point with respect to $\triangle ABC$.

Fact 1.6. E and F lie on the circle with diameter BC (and center M_A).

Fact 1.7. $\triangle AEF \sim \triangle ABC$ and $\triangle BHF \sim \triangle CHE$.

Fact 1.8. O_AOM_AH and AOM_AO_A are parallelograms.

Fact 1.9. E and F lie on the circle with diameter AH (and center O_A).

Fact 1.10. P_A and X_A also lie on this circle.

Fact 1.11. The tangents to the circumcircle of $\triangle AEF$ at points E and F intersect at M_A.

Fact 1.12. $\triangle P_AEF \sim \triangle P_ACB$.

Fact 1.13. A, P_A, and Q_A are collinear.

Fact 1.14. X_A lies on the circumcircle of $\triangle BHC$.

Fact 1.15. $\angle CBX_A = \angle BAX_A$ and $\angle BCX_A = \angle CAX_A$.

Fact 1.16. X_A lies on the A-Apollonius circle (in other words, $\frac{X_AB}{X_AC} = \frac{AB}{AC}$).

Fact 1.17. There are a lot of cyclic quads.

P_A is particulary useful as the center of a spiral similarity. Also, inverting works well with this configuration. Inversion about A with radius $\sqrt{AH \cdot AD}$, about M_A with radius M_AB, and about H with radius $\sqrt{HA \cdot HD}$ are all good options to try.

Problems

Problem 1.1 (IberoAmerican 2011). Let ABC be an acute-angled triangle, with $AC \neq BC$ and let O be its circumcenter. Let P and Q be points such that $BOAP$ and $COPQ$ are parallelograms. Show that Q is the orthocenter of ABC.

Problem 1.2 (USAMO 1990). An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

Problem 1.3 (JBMO 2019). Triangle ABC is such that $AB < AC$. The perpendicular bisector of side BC intersects lines AB and AC at points P and Q, respectively. Let H be the orthocenter of triangle ABC, and let M and N be the midpoints of segments BC and PQ, respectively. Prove that lines HM and AN meet on the circumcircle of ABC.

Problem 1.4 (PAMO 2017). Let ABC be a triangle with H its orthocenter. The circle with diameter AC cuts the circumcircle of triangle ABH at K. Prove that the point of intersection of the lines CK and BH is the midpoint of the segment BH.

Problem 1.5 (USA TSTST 2012). In scalene triangle ABC, let the feet of the perpendiculars from A to BC, B to CA, C to AB be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides BC, CA, AB. Show that the perpendiculars from D to AA_2, E to BB_2 and F to CC_2 are concurrent.

Problem 1.6 (ELMO 2017). Let ABC be a triangle with orthocenter H, and let M be the midpoint of $\overline{BC}$. Suppose that P and Q are distinct points on the circle with diameter $\overline{AH}$, different from A, such that M lies on line PQ. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC$.

Problem 1.7 (Iran TST 2019). Acute-angled triangle ABC has orthocenter H. The reflection of the nine-point circle about AH intersects the circumcircle of $\triangle ABC$ at points X and Y. Prove that AH is the external bisector of $\angle XHY$.

Problem 1.8 (USA TST 2011). Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $OA \perp RA$.

Problem 1.9 (Orthic axis). Let D, E, and F be the feet of the altitudes through A, B, and C respectively in $\triangle ABC$. Let P, Q, and R be the intersections of EF with BC, FD with CA, and DE with AB. Prove that P, Q, and R lie on a line perpendicular to the Euler line.

Problem 1.10 (Iran TST 2011). In acute triangle ABC, $\angle B > \angle C$. Let M be the midpoint of BC. D and E are the feet of the altitudes from C and B respectively. K and L are the midpoints of ME and MD respectively. If KL intersects the line through A parallel to BC at T, prove that $TA = TM$.

Problem 1.11 (APMO 2012). Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of BC, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.

Problem 1.12 (ELMO 2018)**.** Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle of $\triangle ABC$.

Problem 1.13 (Iran MO 2017)**.** Let ABC be an acute-angle triangle. Suppose that M be the midpoint of BC and H be the orthocenter of ABC. Let $E \equiv BH \cap AC$ and $F \equiv CH \cap AB$. Suppose that X be a point on EF such that $\angle XMH = \angle HAM$ and A, X are in the distinct side of MH. Prove that AH bisects MX.

Problem 1.14. In triangle ABC, let A_1, B_1, C_1 be the feet of the altitudes. Let H be the orthocenter. Let M be the midpoint of BC. Let T be the intersection of B_1C_1 and HM. The tangents at B and C to the circumcircle of $\triangle ABC$ intersect at P. Show that T, A_1, P are collinear.

Problem 1.15 (Iran MO 2013)**.** In a triangle ABC with circumcircle Γ, suppose that the A-altitude intersects Γ at point D. The altitude of B and C cut AC and AB at E and F respectively. Let H be the orthocenter and T be the midpoint of AH. The line through T parallel to EF intersects AB and AC at X and Y respectively. Prove that $\angle XDF = \angle YDE$.

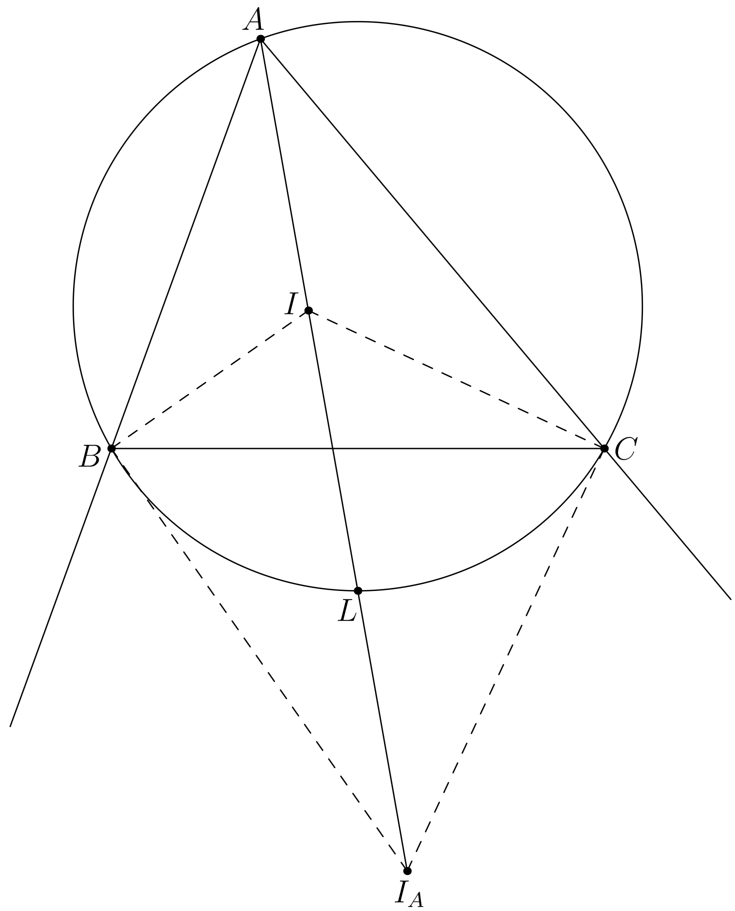
Problem 1.16 (ISL 2008)**.** In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the point A and F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.

Problem 1.17 (ISL 2017)**.** Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

Problem 1.18 (ISL 2016)**.** Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

Problem 1.19 (RMM 2018)**.** Fix a circle Γ, a line ℓ to tangent Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω meet ℓ at Y and Z. Prove that, as X varies over Ω, the circumcircle of XYZ is tangent to two fixed circles.

2 Incenters and Excenters



Let $\triangle ABC$ have incenter I and circumcircle Γ. Denote the excenters as I_A, I_B, and I_C. Let L_A be the midpoint of arc $\widehat{BC}$ not containing A.

Fact 2.1. $\angle BIC = \frac{\pi+\angle A}{2}$.

Lemma 2 (Fact 5)**.** L_A lies on the angle bisector of $\angle A$. Furthermore, $BICI_A$ is a cyclic quadrilateral and L_A is the center of its circumcircle.

Proof. Since L_A is the midpoint of the arc BC, $\angle CAL_A = \angle L_AAB$. So L_A lies on the bisector of $\angle A$. We have

$$\angle IBI_A = \frac{\angle B}{2} + \frac{\pi - \angle B}{2} = \frac{\pi}{2}.$$

Similarly, $\angle ICI_A = \frac{\pi}{2}$. So B and C lie on a circle with diameter II_A.

From cyclic quadrilateral $ABLC$ we have $\angle BL_AI = \angle C$ and from cyclic quadrilateral $BICI_A$ we have $\angle BI_AI = \frac{\angle C}{2}$. Thus L_A must be the center of the circle with diameter II_A. $\square$

Corollary 3. For any $\triangle XYZ$, let W be the intersection of the perpendicular bisector of YZ with the angle bisector of $\angle X$. Then W lies on the circumcircle of $\triangle XYZ$.

Lemma 4 (Euler's Theorem)**.** $OI^2 = R(R - 2r)$ where R is the circumradius of $\triangle ABC$ and r is the inradius of $\triangle ABC$.

Proof. We can compute the distance OI by considering the power of I with respect to Γ:

$$OI^2 - R^2 = Pow(I, \Gamma) = -IA \cdot IL_A = -IA \cdot L_AC.$$

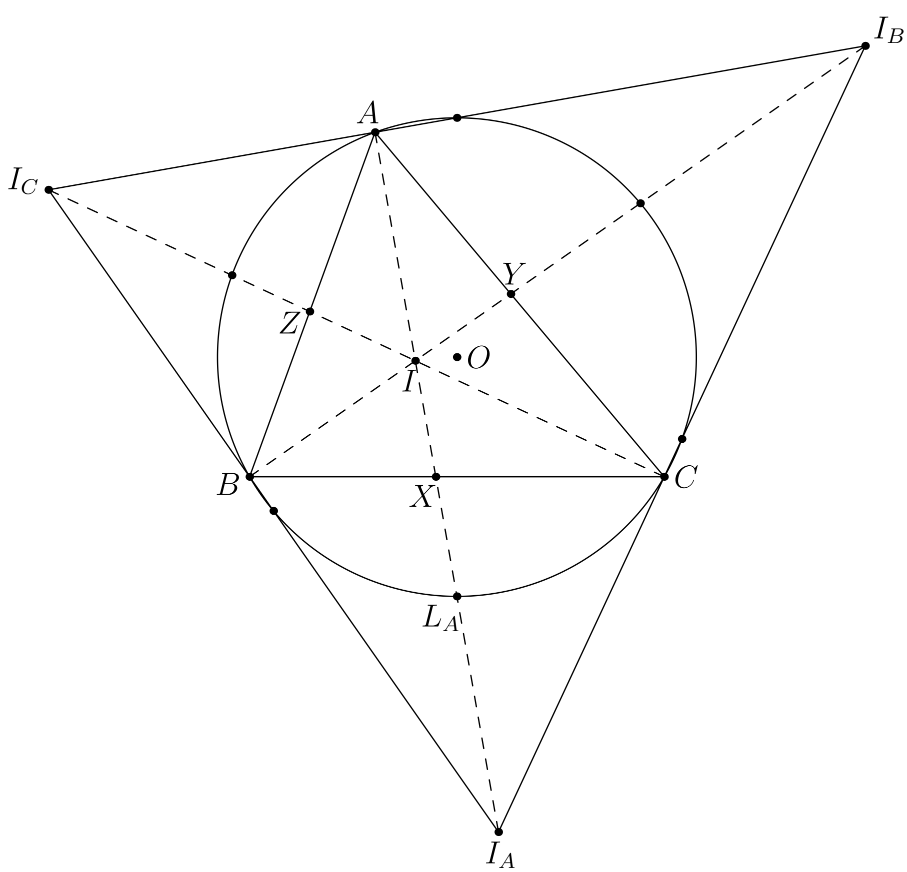
We used Lemma 2 in the above line. Let F be the tangency point of the incircle to AB and let M be the midpoint of L_AC. Note that $\triangle IAD \sim \triangle L_AOM$. Thus

$$\begin{aligned} \frac{AI}{ID} &= \frac{OL_A}{L_AM} \\ \implies \quad IA \cdot L_AM &= ID \cdot OL_A \\ \implies \quad IA \cdot L_AC &= 2ID \cdot OL_A \\ &= 2rR. \end{aligned}$$

Thus,

$$OI^2 = R(R - 2r).$$

□



Let X, Y, and Z be the intersections of the angle bisectors with the respective sides.

Fact 2.2. I is the orthocenter of $\triangle L_AL_BL_C$ (where L_B and L_C are defined similarly).

Fact 2.3. $\triangle L_AXC \sim \triangle L_ACA$ and $L_AX \cdot L_AA = LI^2$.

Fact 2.4. $\triangle ABC$ is the orthic triangle of $\triangle I_AI_BI_C$. I is the orthocenter and Γ is the nine-point circle.

Fact 2.5. The midpoint of arc $\overset{\frown}{BC}$ containing A in Γ is the midpoint of I_BI_C.

Fact 2.6. The radical axis of the B-excircle and C-excircle is the line through the midpoint of BC parallel to AI. The radical axis of the incircle and the A-excircle is the line through the midpoint of BC perpendicular to AI.

Fact 2.7. The intouch triangle, the excentral triangle, and $\triangle L_AL_BL_C$ (where L_B and L_C are defined similarly) are homothetic to each other.

Fact 2.8. The Euler line of the excentral triangle is OI. The Euler line of the intouch triangle is also OI.

Fact 2.9. The circumcenter of $\triangle II_BI_C$ lies on line OI_A.

Fact 2.10. Lines I_BI_C, YZ, and BC concur at the foot of the A-external angle bisector.

Fact 2.11. $YZ \perp OI_A$.

Problems

Problem 2.1. Let $ABCD$ be a cyclic quadrilateral. Let I_A, I_B, I_C, and I_D be the incenters of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$ respectively. Show that $I_AI_BI_CI_D$ is a rectangle.

Problem 2.2. Let ABC be an acute triangle with $\angle A = 60^\circ$. Show that $IH = IO$ where I, H, and O are the incenter, orthocenter, and circumcenter respectively.

Problem 2.3 (Hong Kong TST 2020)**.** Let ΔABC be an acute triangle with incenter I and orthocenter H. AI meets the circumcircle of ΔABC again at M. Suppose the length IM is exactly the circumradius of ΔABC. Show that $AH \geq AI$.

Problem 2.4 (ELMOSL 2013)**.** Let ABC be a triangle with incenter I. Let U, V and W be the intersections of the angle bisectors of angles A, B, and C with the incircle, so that V lies between B and I, and similarly with U and W. Let X, Y, and Z be the points of tangency of the incircle of triangle ABC with BC, AC, and AB, respectively. Let triangle UVW be the *David Yang* triangle of ABC and let XYZ be the *Scott Wu* triangle of ABC. Prove that the David Yang and Scott Wu triangles of a triangle are congruent if and only if ABC is equilateral.

Problem 2.5 (USAJMO 2016)**.** The isosceles triangle $\triangle ABC$, with $AB = AC$, is inscribed in the circle ω. Let P be a variable point on the arc $\overset{\frown}{BC}$ that does not contain A, and let I_B and I_C denote the incenters of triangles $\triangle ABP$ and $\triangle ACP$, respectively. Prove that as P varies, the circumcircle of triangle $\triangle PI_BI_C$ passes through a fixed point.

Problem 2.6 (USAJMO 2014)**.** Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let E, F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc BAC of Γ.

(a) Prove that I lies on ray CV.

(b) Prove that line XI bisects $\overline{UV}$.

Problem 2.7. Let $\triangle ABC$ have incenter I and orthocenter H. Let R_A be the radical center of the incircle, B-excircle, and C-excircle. Define R_B and R_C similarly. Prove that the circumcenter of $\triangle R_A R_B R_C$ is the midpoint of HI.

Problem 2.8 (ISL 2002)**.** The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

Problem 2.9 (USAMO 2017)**.** Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

Problem 2.10 (USAMO 2016)**.** Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$. Lines $\overleftrightarrow{I_B F}$ and $\overleftrightarrow{I_C E}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

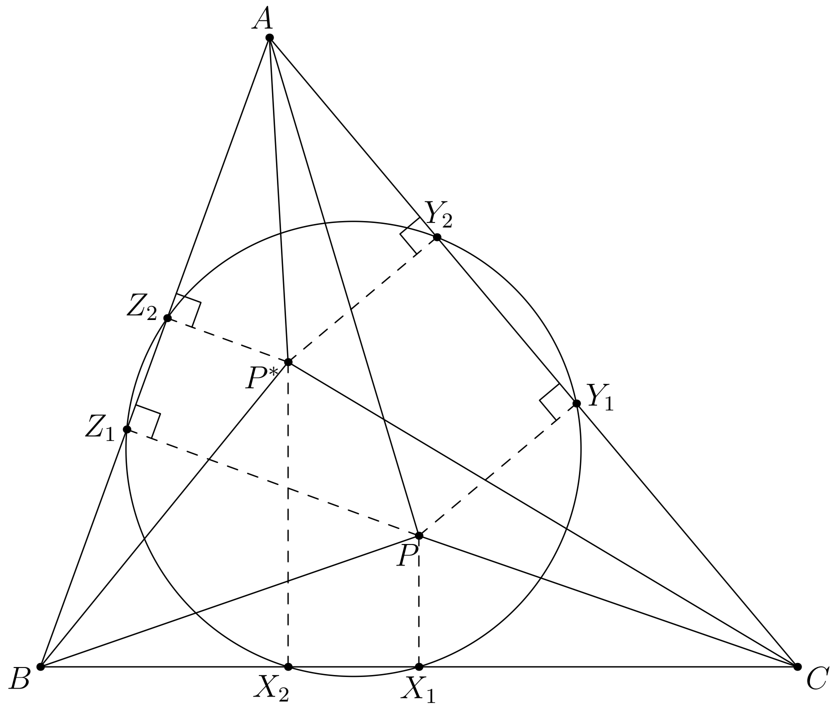
Problem 2.11 (ISL 2016)**.** Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120°$.

Problem 2.12 (Serbia MO 2017)**.** Let k be the circumcircle of $\triangle ABC$ and let k_a be A-excircle. Let the two external tangents of k and k_a cut BC in at points P and Q. Prove that $\measuredangle PAB = \measuredangle CAQ$.

3 Isogonal Conjugates



Lemma 5 (Pedal Triangles of Isogonal Conjugates)**.** Let P_1 and P_2 be points in the plane of $\triangle ABC$. Let $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ be the pedal triangles of P_1 and P_2 respectively. Then P_1 and P_2 are isogonal conjugates iff $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ have the same circumcircle.

Proof. Let P be a point in the plane of $\triangle ABC$ with pedal triangle $\triangle X_1Y_1Z_1$. The circumcircle of $\triangle X_1Y_1Z_1$ intersects BC, CA, and AB again at X_2, Y_2, and Z_2 respectively. Let O be the circumcenter of $\triangle X_1Y_1Z_1$.

Define P^* to be the reflection of P across O. Since O is on the perpendicular bisector of X_1X_2 and P is on the line through X_1 perpendicular to BC, then P^* is on the line through X_2 perpendicular to BC. Similarly, $P^*Y_2 \perp CA$ and $P^*Z_2 \perp AB$.

Since $Y_1Z_1Z_2Y_2$ is a cyclic quadrilateral, $\triangle AY_1Z_1 \sim \triangle AZ_2Y_2$. P and P^* are defined similarly with respect to these two triangles, so we can further state that $AY_1Z_1P \sim AZ_2Y_2P^*$. In particular,

$$\angle PAC = \angle PAY_1 = \angle P^*AZ_2 = \angle P^*AB.$$

Similarly, $\angle PCB = \angle P^*CA$ and $\angle PBC = \angle P^*BA$. Thus, P^* is the isogonal conjugate of P.

This shows one direction of Lemma 5: that if the pedal triangles share a circumcircle, then the points are isogonal conjugates. The definition of the isogonal conjugate of a point implies that there is either none or exactly one. Since we have shown how to construct this isogonal conjugate for any point P, the other direction follows.

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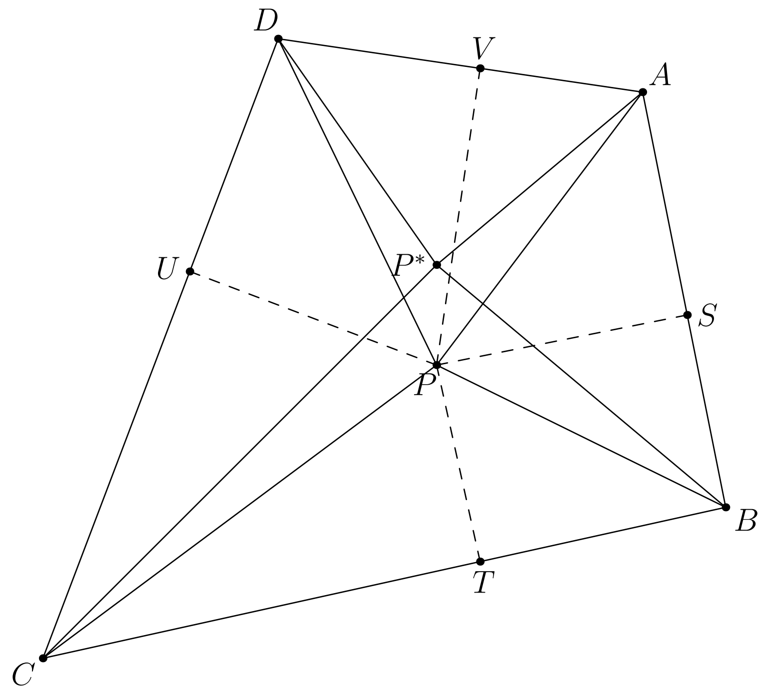

Let P be a point in the plane of quadrilateral $ABCD$. The isogonal conjugate of P with respect to $ABCD$ is the point P^* such that

$$\angle DAP = \angle P^*AB,\ \angle ABP = \angle P^*BC,\ \angle BCP = \angle P^*CD, \text{ and } \angle CDP = \angle P^*DA.$$

Lemma 6 (Isogonal Conjugates in Quadrilaterals)**.** The isogonal conjugate of P with respect to $ABCD$ exists iff $\angle APB + \angle CPD = \pi$.

Proof. Let E be the intersection of AD and BC. Let S, T, U, and V be the feet of P to AB, BC, CD, and DA respectively. Let P_1 be the isogonal conjugate of P with respect to $\triangle EAB$ and P_2 be the isogonal conjugate of P with respect to $\triangle ECD$.

Observe that the isogonal conjugate of P with respect to $ABCD$ exists iff $P_1 \equiv P_2$. It is not hard to see that this occurs iff the circumcircle of the pedal triangle of P_1 with respect to $\triangle EAB$ is the same as the circumcircle of the pedal triangle of P_2 with respect to $\triangle ECD$. By Lemma 5, this is equivalent to S, T, U, V concyclic.

It remains to show that S, T, U, V concyclic iff $\angle APB + \angle CPD = \pi$.

$$\begin{aligned}
\angle APB + \angle CPD &= \angle APS + \angle SPB + \angle CPU + \angle UPD \\
&= \angle AVS + \angle STB + \angle CTU + \angle UVD \\
&= (\pi - \angle STU) + (\pi - \angle UVS) \\
&= 2\pi - (\angle STU + \angle UVS).
\end{aligned}$$

□

Problems

Problem 3.1. Prove that the circumcenter O and the orthocenter H are isogonal conjugates. What is the circumcircle of their pedal triangles?

Problem 3.2. Let P and Q be isogonal conjugates with respect to a triangle ABC. Show that $d(P, AB) \cdot d(Q, AB) = d(P, AC) \cdot d(Q, AC)$.

Problem 3.3. Let P be a point in the interior of $\triangle ABC$. Points D, E, and F are the reflections of P over BC, CA, and AB respectively. Show that the circumcenter of $\triangle DEF$ is the isogonal conjugate of P with respect to $\triangle ABC$.

Problem 3.4 (Bulgaria 2011)**.** Point O is inside $\triangle ABC$. The feet of perpendicular from O to BC, CA, AB are D, E, F. Perpendiculars from A and B respectively to EF and FD meet at P. Let H be the foot of perpendicular from P to AB. Prove that D, E, F, H are concyclic.

Problem 3.5 (USAMO 2011)**.** Let P be a given point inside quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \quad \angle Q_1CB = \angle DCP, \quad \angle Q_2AD = \angle BAP, \quad \angle Q_2DA = \angle CDP.$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

Problem 3.6 (USA TST 2010)**.** In triangle ABC, let P and Q be two interior points such that $\angle ABP = \angle QBC$ and $\angle ACP = \angle QCB$. Point D lies on segment BC. Prove that $\angle APB + \angle DPC = 180^\circ$ if and only if $\angle AQC + \angle DQB = 180^\circ$.

Problem 3.7 (ISL 2008)**.** There is given a convex quadrilateral $ABCD$. Prove that there exists a point P inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$$

if and only if the diagonals AC and BD are perpendicular.

Problem 3.8 (USAJMO 2015)**.** Let $ABCD$ be a quadrilateral. Prove that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if there exists a point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$ and $\angle CDB = \angle YDA$. (The original problem had $ABCD$ cyclic.)

Problem 3.9 (IMO 2018)**.** A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180^\circ$.

Problem 3.10 (ELMOSL 2014)**.** We are given triangles ABC and DEF such that $D \in BC, E \in CA, F \in AB$, $AD \perp EF, BE \perp FD, CF \perp DE$. Let the circumcenter of DEF be O, and let the circumcircle of DEF intersect BC, CA, AB again at R, S, T respectively. Prove that the perpendiculars to BC, CA, AB through D, E, F respectively intersect at a point X, and the lines AR, BS, CT intersect at a point Y, such that O, X, Y are collinear.

Problem 3.11 (USA TST 2015)**.** Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.