Graphs, Graphs, and Graphs

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1 Introduction

A graph $G := (V, E)$ is a collection of vertices V and edges E. In this handout, we will primarily discuss *undirected* graphs in which edges are bidirectional. The *degree* of a vertex is the number of edges incident to it.

Theorem 1 (Handshaking Lemma)**.** The sum of degrees is equal to twice the number of edges.

Proof. Let X be the number of vertex-edge pairs (v, e) where e is incident to v. We can count this either based on each vertex v, or each edge e. For each vertex v, there are $\deg(v)$ edges incident to v. On the other hand, for each edge e, there are two vertices incident to e. Hence,

$$X = \sum_{v \in V} \deg(v) = 2|E|. \qquad \square$$

This lemma is useful for keeping track of parities. We will see how this can be used to prove existence in the following example, the two-dimensional case of Sperner's Lemma.

Example 1 (Sperner's Lemma)**.** Let T be a triangle $P_1P_2P_3$ and $\mathcal{T}$ a triangulation of T. We define a proper colouring of $\mathcal{T}$ as an assignment of 3 colours to the vertices of $\mathcal{T}$ such that

(i) The colours of P_1, P_2, P_3 are distinct;

(ii) The vertices on the edges of T, P_iP_j, have the same colour as P_i or P_j.

Then any proper coloured triangulation must have a triangle whose vertices have all different colours.

Proof. Consider the graph whose vertices correspond to the faces of the triangulation, as well as one vertex for the area outside the triangle. We draw an edge between two vertices if they share a blue-green edge in the triangulation. (This type of structure is known as a dual graph.) The degrees of the internal vertices must be $0, 1$, or 2. In addition, if an internal vertex has degree 1, then its corresponding face must be tri-coloured.

Thus, it suffices to prove that there exists an internal vertex with degree 1. By the Handshaking Lemma, the total of the degrees is even. Note that the degree of the external vertex must be odd thanks to condition (ii) of the proper colouring. So we must have an odd number of internal vertices with odd degree and so we are done. $\square$

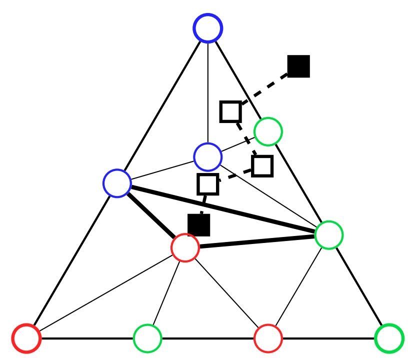

Figure 1. Sperner's Lemma in 2D and the dual graph used in the proof.

From the concept of vertices and edges alone, we're already able to prove some interesting statements. In the rest of this section, we'll familiarize ourselves with common graphs and graph structures.

1.1 Trees

A tree is, in a sense, a sparse sort of graph. Before introducing trees formally, let's define a few terms. A *path* is a sequence of distinct edges which join a sequence of distinct vertices. A *cycle* is a path except it starts and ends at the same vertex. A graph is *connected* if there is a path between every pair of vertices.

Definition. An undirected graph G is a *tree* if it is connected and acyclic (contains no cycles).

Theorem 2. Let G be an undirected graph with n vertices. Then the following conditions are all equivalent:

(i) G is connected and acyclic

(ii) G is connected and has $n-1$ edges

(iii) Any two vertices of G are connected by a unique path

Proof. We will first prove (i) $\iff$ (iii). It's clear that the condition that G is connected is equivalent to any two vertices being connected by at least one path. We need to show that the acyclic condition is equivalent to the uniqueness of all paths. We will show the complement: the existence of a cycle is equivalent to the existence of two vertices with multiple paths connecting them. If G has a cycle, then vertices in that cycle will have at least two paths connecting them. Now if there exists a pair of vertices with two paths between them, consider the pair (u, v) for which the two paths have minimal total length. By minimality, the union of these two paths must be a cycle without repeated vertices. So we have proved (i) $\iff$ (iii).

A *leaf* is a vertex with degree 1. Recall from the outside world that trees have leaves. We will prove, as a corollary of (iii), that trees with at least two vertices indeed have leaves. Consider a path in the tree with maximal length, known as a *diameter*. Assume for the sake of contradiction that one of the endpoints of the diameter has degree > 1. By the maximality assumption, its neighbours must be in the path. However, as the tree is acyclic, only one neighbour can be in the path. This is a contradiction so the endpoints of the diameter are leaves.

The existence of leaves is useful for induction, as we now know that we can go from a tree of size n to a tree of size $n-1$ by detaching a leaf. Let's prove (i) $\iff$ (ii) with this idea in mind. The base case of $n = 1$ is clear. Now assume that we have proved (i) $\iff$ (ii) for all graphs G with n vertices.

Let's first prove (i) $\implies$ (ii) for $n+1$. If G is a tree with $n+1$ vertices, remove any leaf to get a tree G'. By the hypothesis for n, G' has $n-1$ edges and so G must have had n edges.

Now let's prove (ii) $\implies$ (i). Say G is a connected graph with $n+1$ vertices and n edges. We will prove that G has a vertex of degree 1. As G is connected, all vertices have degree ≥ 1. Assume for the sake of contradiction that there is no vertex with degree 1. By the Handshaking Lemma, the total of the $n+1$ degrees is $2n$. But this is impossible if the degrees are all at least 2 – hence, there is a contradiction and there exists some vertex of degree 1. Now we can remove this vertex and induct as we did before. So (i) $\iff$ (ii) is proved, and we are done. □

Lemma 1. A graph G has n vertices and m edges. If $m \geq n$, then G has a cycle.

Proof. Write G as the disjoint union of its connected components $(V_1, E_1), (V_2, E_2), \ldots, (V_k, E_k)$. Note that

$$m = \sum_{i=1}^{k} |E_i| \geq \sum_{i=1}^{k} |V_i| = n.$$

So we can choose some connected component i for which $|E_i| \geq |V_i|$. This subgraph is connected, so we can apply Theorem 2 to see that it is not acyclic. In other words, this component has a cycle. □

Example 2. N ants are placed on the edges of a unit cube. Find the minimal N such that there must exist two ants that are at a walking distance at most 1 from each other. Assume that ants can walk only along the edges.

Proof. Consider the graph corresponding to the eight vertices of the unit cube and its twelve edges. The problem's condition is equivalent to the ants' edges forming a cycle. Indeed, if we have k ants on a cycle of size k, the total distance between each pair of the ants is k. Thus, there must be some pair within a distance of 1. If no cycle exists, it's not hard to see that we can put the ants in the centers of the edges with arbitrarily small shifts.

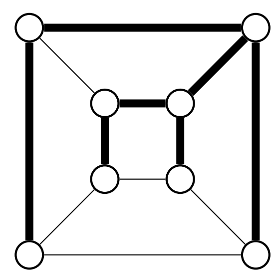

Figure 2. Tree with seven edges.

Now returning to the graph in question, we claim that the answer is $N = 8$, the number of vertices in the graph. If eight ants are placed, Lemma 1 tells us that there exists a cycle formed by a subset of their edges. It is easy to find constructions for seven ants with no ants with a distance of 1. □

Every connected graph G has at least one tree containing all vertices as a subgraph. Such a tree is known as a *spanning tree*. In the previous example, there were many arrangements with seven ants corresponding to a spanning tree. The existence of many equality cases which are difficult to pin down often hints that trees may be at play.

Theorem 3 (Cayley's Formula)**.** The complete graph of n vertices has n^{n-2} spanning trees.

We will omit the proof, but there is a very elegant bijection proof involving Prüfer codes. There also exists the more general formula, Kirchhoff's Matrix Tree Theorem, which gives the number of spanning trees of a graph G with n vertices as the determinant of an $(n-1) \times (n-1)$ matrix.

1.2 Bipartite Graphs

Definition. A graph $G := (V, E)$ is *bipartite* if its vertex set V can be partitioned into two non-empty sets X, Y such that there are no edges between vertices of the same set.

Bipartite graphs can appear in grid problems by forming a graph with rows and columns as the sets X, Y and cells as edges between the corresponding row and column. They are also integral to matching and assignment problems. Before we discuss these applications in more detail, let's review the following characterization of bipartite graphs.

Lemma 2. A graph G is bipartite iff it contains no cycles of odd length.

Proof. One direction is clear. If G has a cycle of odd length, it is impossible to partition the vertices of this cycle. Now assume that G does not have any cycle of odd length. We can colour the vertices of G black or white iteratively, so that no two adjacent vertices are the same colour. As there are no odd cycles, we can do this for the entire graph. The sets of vertices of each colour form the partition. □

Example 3 (IMC 1999)**.** Suppose that $2n$ points of an $n \times n$ grid are marked. Show that for some $k > 1$ one can select $2k$ distinct marked points, say $a_1, ..., a_{2k}$, such that a_{2i-1} and a_{2i} are in the same row, a_{2i} and a_{2i+1} are in the same column, $\forall i$, indices taken mod $2n$.

Proof. We will construct a bipartite graph from a marked grid. The vertices are the rows $r_1, \ldots, r_n$ and columns $c_1, \ldots, c_n$ of the grid. If cell (i, j) is marked, then we draw an edge between r_i and c_j. We have $2n$ vertices and $2n$ edges, so by Lemma 1 and the fact that this is a bipartite graph, there exists a cycle of length $2k$ for $k > 1$. The cells corresponding to the cycle's edges satisfy the desired conditions. □

Definition. A *matching* in a graph G is a subset of the edges which do not have any common vertices. A *perfect matching* covers every vertex.

Theorem 4 (Hall's Marriage Theorem)**.** Let G be a bipartite graph with vertices V partitioned as $X \sqcup Y$ and edges E. For a set of vertices W, let the neighbourhood of W, $N_G(W)$, denote the set of vertices adjacent to any vertex in W. Then there exists a matching which covers X iff

$$|W| \leq |N_G(W)|$$

for all subsets W of X.

Proof. First, if there does exist an X-perfect-matching, then the neighbourhood of any $W \subseteq X$ will include the $|W|$ corresponding matched elements of Y and so the inequalities clearly hold.

To prove the other direction, we will use strong induction on the size of X. When X has a single vertex, the result is obvious. Now say $|X| \geq 2$. Consider an arbitrary edge (x, y) with $x \in X$ and $y \in Y$. If the inequalities hold for the induced subgraph with vertices $(X \setminus \{x\}) \cup (Y \setminus \{y\})$ then we can match x and y and induct for the rest.

Otherwise, there is some set $W \subseteq X - \{x\}$ for which $y \in N_G(W)$ and $|W| = |N_G(W)|$. As $|W| < |X|$ and the neighbourhood inequalities hold, our induction hypothesis tells us that there exists a matching from W to $N_G(W)$. Let G' denote the graph induced by vertices $X \setminus W$ and $Y \setminus N_G(W)$. If we can prove the neighbourhood inequalities for G', then we can find a matching by induction and finish. Consider arbitrary $U \subseteq X \setminus W$. We know from the original G that

$$\begin{aligned} |U \cup W| &\leq |N_G(U \cup W)| \\ |U| + |W| &\leq |N_{G'}(U)| + |N_G(W)| \\ |U| &\leq |N_{G'}(U)|. \end{aligned}$$

So the inequalities indeed hold and there is a matching of G' that covers $X \setminus W$. By combining this matching with the matching from W to $N_G(W)$, we have a matching of G that covers X. By induction, the theorem is proved for all bipartite graphs G.

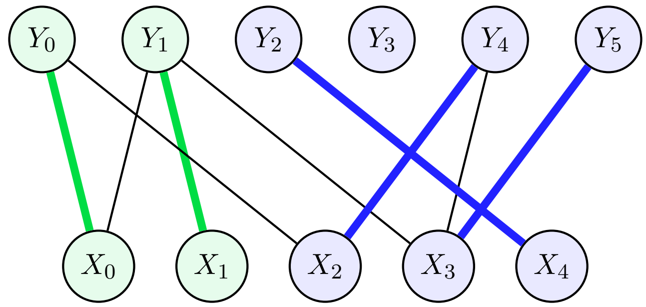

Figure 3. An example of the induction step with $W = \{X_0, X_1\}, N_G(W) = \{Y_0, Y_1\}$ in light green, and the rest of the graph in light blue. The matching formed between W and $N_G(W)$ is in bolded green while the matching formed within G' is in bolded blue. □

Lemma 3. Let k be a positive integer and G be a bipartite graph such that every vertex has degree k. Then the edges of G can be partitioned into k perfect matchings.

Proof. Denote the bipartite sets as X, Y. Note that the number of edges is $k|X| = k|Y|$, so the two sets are of the same size. We will first prove that G must have a perfect matching. Let W be a subset of X. It suffices to verify the neighbourhood inequalities. Note that there are exactly $k|W|$ edges between W and $N_G(W)$. But we can also see that this number of edges is at most $k|N_G(W)|$. Hence, $|W| \leq |N_G(W)|$, as desired. By Hall's Marriage Theorem, there is a perfect matching. Now by removing the edges in this perfect matching, we get a new graph in which all vertices have degree $k-1$ and we can repeat to partition all edges into disjoint perfect matchings. □

1.3 Eulerian Graphs

In the following discussion, we will use *trails* and *tours* rather than paths and cycles. The only distinction is that trails and tours may visit vertices multiple times. Sometimes, the terms are interchanged anyways.

Theorem 5 (Eulerian Tours)**.** In a connected graph, an Eulerian tour is defined as a tour that uses every edge exactly once and starts and ends at the same vertex. A connected graph has an Eulerian tour iff the degree of every vertex is even.

Eulerian trails are defined similarly, except without the condition that the starting and ending vertices are the same. A graph is Eulerian if it has an Eulerian tour.

Proof. We will first prove that the existence of an Eulerian tour implies even degrees. For a given vertex v, consider the $\deg(v)$ edges incident to v. Each appears exactly once in the Euler tour. These edges also appear in pairs: one when the tour enters v and one when it exits. So $\deg(v)$ must be even.

Now we will prove that even degrees and connectedness implies an Eulerian graph. We will use strong induction on the number of edges. Consider G and observe that G contains some cycle C. This is immediate from the fact that all degrees in G are ≥ 2. There is also an algorithmic way of constructing a cycle in a graph of even degrees. You can choose an arbitrary vertex and keep walking along unused edges. By parity, any vertex other than the starting one has an odd number of unused edges when it's reached, so this walk is guaranteed to return to the starting vertex.

Consider the graph $G \setminus C$, which also has all degrees of even parity. Let $G_1, G_2, \ldots, G_k$ be the connected components of $G \setminus C$. By induction, each G_i has an Eulerian tour. In addition, since the original graph G was connected, each connected component shares at least one vertex v_i with C. We can build a full Eulerian tour of G by walking along C, and when we reach a vertex v_i, following the Eulerian tour for G_i, starting at and returning to v_i. So G is Eulerian and by induction, we've proved that all connected graphs with even degrees are Eulerian. □

Lemma 4. Let G be a graph that has $2k$ vertices with odd degree. Assume that the edges of G can be covered by m trails without overlap. Then $m \geq k$.

Proof. Let u_i, v_i be the starting and ending vertices of the ith trail. Consider the graph G' obtained by adding the m edges (v_i, u_{i+1}) to G, with indices taking mod m. Then the m trails connected by these edges are an Eulerian tour of G', and hence G' should have all even degrees. Note that G' has at most $2m$ fewer vertices with odd degree compared to G, and so $m \geq k$. □

1.4 Hamiltonian Graphs

Definition. A Hamiltonian path of G is a path which uses every vertex of G. A Hamiltonian cycle of G is a cycle which uses every vertex of G.

A graph is Hamiltonian if it contains a Hamiltonian cycle. Though the definition is (superficially) similar to that of Eulerian graphs, Hamiltonian graphs are far more cumbersome to analyze. As a result, they appear infrequently in olympiads, and ad-hoc approaches tend to be more common than appealing to theory. Regardless, it's good to learn new things.

Theorem 6 (Hamiltonian Decomposition)**.** Let K_n be the complete graph of n vertices. If n is odd, its edges can be partitioned into $\frac{n-1}{2}$ Hamiltonian cycles.

We won't prove this but we give an example in the right figure.

The Bondy-Chvátal Theorem, which we state without proof below, gives some direction to characterizing Hamiltonian graphs. Before stating the theorem, we must first define a graph *closure*. The closure of a graph G with n vertices is obtained by repeatedly adding an edge between non-adjacent vertices u and v with $\deg(u) + \deg(v) \geq n$, until no more such pairs u, v exist.

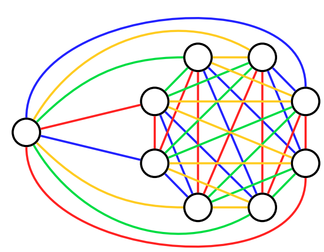

Figure 4. $n = 9$ construction.

Theorem 7 (Bondy-Chvátal Theorem)**.** A graph is Hamiltonian iff its closure is Hamiltonian.

As a corollary of the Bondy-Chvátal Theorem, we have the slightly more direct Dirac's Theorem.

Theorem 8 (Dirac's Theorem)**.** Let G be a graph with n vertices. If every vertex has degree $\geq \frac{n}{2}$, then G is Hamiltonian.

We'll close off this section with a result on Hamiltonian paths in directed graphs.

Example 4 (Redei)**.** A tournament is held with $n \geq 2$ players, where every pair of players faces off once in a match, resulting in a winner and a loser. Prove that the players can be ordered from 1 to n so that person $i+1$ won against person i.

Proof. We will prove this statement by induction. For the base case of $n = 2$, the ordering is obvious. Now we will show that the statement is true for a tournament of $n+1$ players, assuming it holds for any tournament with n players. Choose an arbitrary player, say Alice. By the hypothesis, the other n players can be ordered 1 through n so that each player beats the previous one. If Alice beat all n players, then we simply let her be person $n+1$. Otherwise, let i be the minimal index such that Alice lost to player i. Then Alice must have beaten player $i-1$, so we can insert Alice between i and $i-1$. This gives a Hamiltonian path, so we are done. □

1.5 Problems

1-1. Prove the following facts about trees:

(i) Every connected graph has a spanning tree.

(ii) A connected graph G has exactly one spanning tree. Prove that G is a tree.

(iii) Let G be an acyclic graph (commonly known as a *forest*) with n vertices and m edges. Prove that G has $n-m$ connected components, each of which are trees.

1-2. (CMO 2006). In a rectangular array of nonnegative reals with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

1-3. (CMO 2018). Two positive integers a and b are prime-related if $a = pb$ or $b = pa$ for some prime p. Find all positive integers n, such that n has at least three divisors, and all the divisors can be arranged without repetition in a circle so that any two adjacent divisors are prime-related. Note that 1 and n are included as divisors.

1-4. In a school, there are three classes each containing M students. Every student is friends with more than $\frac{3M}{4}$ people in each of the other two classes. Prove that the students can be split into M teams of three friends from different classes.

1-5. (De Bruijn). Prove that one can write 2^n numbers around a circle, each equal to 0 or 1 so that any string of n 0's and 1's can be obtained by starting somewhere on the circle and reading the next n digits in clockwise order.

1-6. (ISL 2023). Let $n \geqslant 2$ be a positive integer. Paul has a $1 \times n^2$ rectangular strip consisting of n^2 unit squares, where the i^{th} square is labelled with i for all $1 \leqslant i \leqslant n^2$. He wishes to cut the strip into several pieces, where each piece consists of a number of consecutive unit squares, and then translate (without rotating or flipping) the pieces to obtain an $n \times n$ square satisfying the following property: if the unit square in the i^{th} row and j^{th} column is labelled with a_{ij}, then $a_{ij} - (i+j-1)$ is divisible by n. Determine the smallest number of pieces Paul needs to make in order to accomplish this.

2 Combinatorial Techniques

Let's now discuss common combinatorial techniques to tackle graph theory problems. Specifically, we will discuss the extremal principle, induction, and algorithms. Note that these three techniques are equivalent in many situations. For example, induction on the number of vertices can be rephrased as an algorithm decreasing the number of vertices, or an extremal proof by contradiction that analyzes a contradictory graph with the minimal size. However, for complex proofs, it can be cumbersome to interchange these. The examples below highlight cases where the demonstrated technique is especially useful.

2.1 Extremal Principle

In graph theory problems, objects with minimal or maximal properties can often be exploited. Common candidates include vertices with extreme degree or paths or cycles with extreme lengths. We will look at a more elaborate maximal structure in the following example.

Example 5 (ToT 2009)**.** Anna and Ben are visiting an archipelago with 2009 islands. Some pairs of islands are connected by boats which run both ways. Anna and Ben play the following game: Anna chooses the first island on which they arrive by plane. Then Ben chooses the next island which they could visit. Thereafter, the two take turns choosing an island which they have not yet visited. When they arrive at an island which is connected only to islands they had already visited, whoever's turn to choose next would be the loser. Prove that Anna can always win, regardless of the way Ben plays and regardless of the way the islands are connected.

Proof. Let G be the graph of 2009 vertices and let M be a maximal matching of G. The cardinality of M is even so there must be some vertex v_0 of G not in M. Anna wins by first choosing v_0. We claim that each pair of Ben and Anna's moves will be a matched pair in M until Ben finally loses. On the first turn, say Ben moves to a neighbour u_1 of v_0. Assume for the sake of contradiction that $u_1 \notin M$. This contradicts the maximality of M as then we could add $\{u_1, v_0\}$ into M. So u_1 must be in M and Anna can follow-up with u_1's partner, v_1. As long as Ben keeps moving in M, this will continue, and Anna can keep responding with the corresponding neighbour in M.

Assume for the sake of contradiction that on Ben's $(k+1)$th turn (with k minimal), he moves to a vertex u_{k+1} not in M while all previous pairs u_i, v_i have been in M. Then consider the path $v_0, u_1, v_1, \ldots, v_k, u_{k+1}$. This path can be organized into a matching with $k+1$ pairs, but only k pairs are in M. Again by the maximality of M, we have a contradiction, and so Ben must keep playing into M and Anna can always respond. As the game can only go on for a finite number of turns, Ben eventually loses and Anna wins.

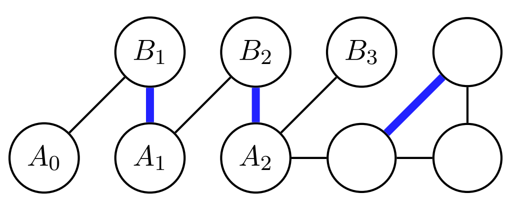

Figure 5. An example sequence for the contradictory case of Example 5, where the blue bolded edges denote the edges of the supposed maximal matching. □

2.2 Induction

Induction is always a reliable way to approach graph theory problems. This is particularly true if the problem is about a family of graphs which is easy to induct on (either by removing vertices or edges), such as a trees or a general graph.

Example 6 (CMO 2010)**.** Each vertex of a graph can be coloured either black or white. Initially all vertices are black. We are allowed to pick a vertex P and change the colour of P and all of its neighbours. Is it possible to change the colour of every vertex from black to white by a sequence of operations of this type?

Proof. It is always possible. We will prove this by induction on the number of vertices of the graph. The base case of $|V| = 1$ is obvious. Now consider a graph G with n vertices. For any vertex v, our inductive hypothesis for $n-1$ tells us that there is a series of operations on $G \setminus \{v\}$ that switches their colours. If there exists a vertex v where we do this and v is white after these operations, we are done.

Otherwise, we now have a *superoperation* on any desired vertex v that switches the colours of all other vertices in G while leaving v unchanged. If n is even, we can apply the superoperation to every vertex in v. This switches the colour of each vertex an odd number $(n-1)$ of times, so we are done. If n is odd, there exists some vertex v which has even degree. We can apply the superoperation to v and all of its neighbours. This leaves v and its neighbours unchanged, but switches the colours of all other vertices. Finally, we perform the original operation on v and we are done. □

2.3 Algorithms

Algorithms are excellent for construction. Often, problems which aren't originally framed as a construction problem can be massaged into one. The following example demonstrates this idea.

Definition. A *k-clique* of a graph G is a subgraph of k vertices which are all pairwise directly connected.

Example 7 (Zarankiewicz)**.** Let G be a graph with n vertices and does not contain any k-clique. Prove that there exists a vertex of degree $\leq \frac{n(k-2)}{(k-1)}$.

Proof. Assume for the sake of contradiction that every vertex in G has degree $> \frac{n(k-2)}{(k-1)}$. We will iteratively construct a cliques C_j of size j from 1 to k. We start with C_1, which contains a single arbitrary vertex v_1. Let P_i be the set of vertices adjacent to every vertex in C_i. If P_i is non-empty, we can add an arbitrary vertex v_{i+1} from P_i to form $C_{i+1} := C_i \cup \{v_{i+1}\}$. We have

$|P_1| = \deg(v_1) > \frac{n(k-2)}{(k-1)}$. Note that $P_j = \left|\bigcap_{i=1}^{j} N_G(v_i)\right|$. For $j > 1$,

$$\begin{aligned} |P_j| &= |P_{j-1} \cap N_G(v_j)| \\ &= |P_{j-1}| + |N_G(v_j)| - |P_{j-1} \cup N_G(v_j)| \\ &> |P_{j-1}| + \frac{n(k-2)}{(k-1)} - n \\ |P_j| - |P_{j-1}| &> -\frac{n}{k-1}. \end{aligned}$$

By taking the telescoping sum, we see that $|P_j| > \frac{n(k-j-1)}{(k-1)}$. So for $j \le k-1$, P_j is non-empty and so we are able to construct clique C_k of size k. □

2.4 Problems

2-1. Let G be a graph in which every vertex has degree $\ge \delta$ where $\delta > 1$. Prove that G has a cycle of length $\ge \delta + 1$.

2-2. (Russia 2001). In a party, there are $2n+1$ people. For every group of n people, there is a person (not in this group) who knows all n of them. Show that there exists a person who knows everyone in the party.

2-3. Let T be a tree with n vertices and let G be a graph for which every vertex has degree at least $n-1$. Prove that G has a subgraph isomorphic to T.

2-4. Show that every graph G with average degree d has a subgraph in which every vertex has degree at least $d/2$.

2-5. (HMMT 2018). Evan has a simple graph with v vertices and e edges. Show that he can delete at least $\frac{e-v+1}{2}$ edges so that each vertex still has at least half of its original degree.

2-6. (ISL 2013). A crazy physicist discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(ii) At any moment, he may double the whole family of imons in the lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

3 Algebraic Techniques

3.1 Double Counting

A frequent pattern in graph theory problems is to count the number of a given object by summing over individual vertices. The most common set-up is when the quantity for each vertex is a function of the degree, particularly a binomial coefficient. In this situations, we can appeal to the convexity of these functions to give us clean bounds.

Theorem 9 (Jensen's Inequality)**.** Let f be a convex function over interval $\mathcal{I}$. For $x_1, \ldots, x_n \in \mathcal{I}$, we have

$$\frac{\sum_{i=1}^n f(x_i)}{n} \geq f\left(\frac{\sum_{i=1}^n x_i}{n}\right).$$

Lemma 5. For fixed $k \in \mathbb{Z}_{\geq 0}$, the binomial coefficient function $\binom{x}{k}$ is convex for $x \geq k-1$.

Note that $x < k-1$ is technically possible but not an issue for convexity, as we can simply define the function to be 0 for $x \leq k-1$. It's easy to see that this modified function is convex over all the reals.

Example 8. Let G be a graph with n vertices such that there is no 4-cycle. Prove that G has at most $\frac{n+n\sqrt{4n-3}}{4}$ edges.

Proof. Let m be the number of edges. Let X be the number of triplets of vertices (u, v, w) where u and w are neighbours of v, and u, w are unordered. As there are no 4-cycles, any pair (u, w) will have at most one v neighbouring both and so $X \leq \binom{n}{2}$. However, we can also calculate X by first fixing v and choosing among its neighbours. Doing this yields

$$X = \sum_{v \in V} \binom{\deg(v)}{2} \leq \binom{n}{2}.$$

By Jensen's Inequality,

$$\begin{aligned}\binom{n}{2} &\geq \sum_{v \in V} \binom{\deg(v)}{2} \\ &\geq n\binom{\frac{2m}{n}}{2}\end{aligned}$$

Expanding yields a quadratic which we can solve,

$$\begin{aligned}4m^2 - 2mn - n^2(n-1) &\leq 0 \\ m &\leq \frac{2n + \sqrt{(2n)^2 + 16n^2(n-1)}}{8} \\ &= \frac{n + n\sqrt{4n-3}}{4}.\end{aligned}$$

□

In this next example, we will force a graph theory problem to become an algebra problem by using edge indicator variables to represent the graph.

Definition. Let G be a graph with vertices V. A *dominating set* of G is a subset $S \subseteq V$ such that every vertex $v \in V$ is either in S or has a neighbour in S.

Example 9 (USATST 2010)**.** Let T be a finite set of positive integers greater than 1. A subset S of T is called good if for every $t \in T$ there exists some $s \in S$ with $\gcd(s,t) > 1$. Prove that the number of good subsets of T is odd.

Proof. Let G be the graph whose vertex set is T, and for which an edge is drawn between two elements s, t if $\gcd(s,t) > 1$. Then the problem is asking us to show that the number of dominating sets of G is odd. In fact, we will show that the number of dominating sets of any graph is odd. Let e_{ij} be the edge indicator variable such that

$$e_{ij} := \begin{cases} 1 \text{ if } (i,j) \text{ is an edge,} \\ 0 \text{ otherwise.} \end{cases}$$

For a set of vertices $A \subseteq T$, let $D(A) := \prod_{i \in T \setminus A} \left(1 - \prod_{j \in A} (1 - e_{ij})\right)$. (Empty products, e.g. when $A = \emptyset$, evaluate to 1.) This is the indicator function for whether A is a dominating set – if A is, then $D(A)$ is 1, otherwise $D(A)$ is 0. Note that

$$\begin{aligned} D(A) &= \prod_{i \in T \setminus A} \left(1 - \prod_{j \in A} (1 - e_{ij})\right) \\ &\equiv \sum_{B \subseteq T \setminus A} \left(\prod_{i \in B} \prod_{j \in A} (1 - e_{ij})\right) \pmod 2. \end{aligned}$$

The number of dominating sets is

$$\sum_{A \subseteq T} D(A) \equiv \sum_{A \subseteq T} \sum_{B \subseteq T \setminus A} \left(\prod_{i \in B} \prod_{j \in A} (1 - e_{ij})\right) \pmod 2.$$

This sum is completely symmetric between A and B so everything cancels out modulo 2, except when $A = B$. As A and B are disjoint, this only happens when $A = B = \emptyset$, for which the product is empty and evaluates to 1. Hence, the number of dominating sets is odd, as desired. □

Remark. The product $\prod_{i \in B} \prod_{j \in A} (1 - e_{ij})$ is the indicator function for whether there are no edges between A and B. In other words, we were actually counting the number of disjoint subsets A, B with no edges between them. This solution could have been written directly from this observation and without the use of indicator variables. I chose to present it this way as this quantity is hard to motivate otherwise and indicator variables tend to be useful in general.

3.2 Probabilistic Method

The probabilistic method is another technique that can be used to obtain global bounds on local graph structures. For further reading into the probabilistic method in math olympiads, check out Evan Chen's handout.

Definition. An *independent set* of a graph G is a subset of its vertices with no two adjacent.

Theorem 10 (Caro-Wei)**.** Let G be a graph with n vertices whose degrees are $d_1, \ldots, d_n$. Then G has an independent set of size at least

$$\frac{1}{d_1+1} + \cdots + \frac{1}{d_n+1}.$$

Proof. Consider a random permutation of the vertices. We can construct an independent set I from any permutation by iterating through and selecting vertex i if it appears before any of its d_i neighbours. The probability that vertex i is chosen is $\frac{1}{d_i+1}$. By Linearity of Expected Values,

$$\mathbb{E}[|I|] = \sum_i \Pr[i \in I] = \sum_{i=1}^n \frac{1}{d_i+1}.$$

So there exists some independent set at least this size. □

Theorem 10 can be used to directly prove a famous theorem on cliques in graphs, with the observation that independent sets are the complement of cliques. Note that Theorem 10 also proves Example 7.

Theorem 11 (Turán's Theorem)**.** Let G be a graph with n vertices and does not contain any $(r+1)$-clique. Then G has at most $\left(1-\frac{1}{r}\right)\frac{n^2}{2}$ edges.

Proof. Let r be the size of the largest clique in G and m the number of edges of G. Theorem 10 applied to the complement of G tells us that G has a clique of size at least $\sum_{i=1}^n \frac{1}{n-d_i}$. From this, and the Cauchy-Schwarz Inequality, we have

$$r \geq \sum_{i=1}^n \frac{1}{n-d_i} \geq \frac{n^2}{n^2 - \sum_{i=1}^n d_i} = \frac{n^2}{n^2-2m}.$$

Rearranging directly gives $\left(1-\frac{1}{r}\right)\frac{n^2}{2}$. □

Remark. This bound is only sharp when n is divisible by r. A slightly more intricate formula exists for $r \nmid n$, but generally this upper bound suffices. When $n = rt$ for $t \in \mathbb{N}$, the construction that achieves equality is a complete r-partite graph where each of the sets has size t. For example, a triangle-free graph with $2t$ vertices and t^2 edges is possible with $K_{t,t}$.

This last example is very tricky to analyze locally, but by considering probabilistic ideas, a surprisingly clean solution emerges.

Example 10. A graph has n vertices and m edges. If the edges are assigned the labels $1, 2, \ldots, m$, prove that there exists a path consisting of at least $\frac{2m}{n}$ edges such that the labels of the edges along the path are in increasing order.

Proof. Place n ants on the graph, one at each vertex. We will step from 1 to m. When we are at step i, the two ants incident to the edge labelled i will swap locations. Note that after each step, there is exactly one ant at each vertex. After all the steps are done, each ant has walked along a path whose edge labels are in increasing order. Furthermore, the total of the lengths is $2m$ since at each step, the total increased by 2. Thus, there is some ant who walked at least $\frac{2m}{n}$ edges. □

3.3 Problems

3-1. (CMO 2023). There are 20 students in a high school class, and each student has exactly three close friends in the class. Five of the students have bought tickets to an upcoming concert. If any student sees that at least two of their close friends have bought tickets, then they will buy a ticket too. Is it possible that the entire class buys tickets to the concert?

3-2. Given n points in the plane, prove that the number of pairs of points that have distance of length 1 is at most $n^{\frac{3}{2}}$.

3-3. (USATST 2013). A social club has $2k+1$ members, each of whom is fluent in the same k languages. Any pair of members always talk to each other in only one language. Suppose that there were no three members such that they use only one language among them. Let A be the number of three-member subsets such that the three distinct pairs among them use different languages. Find the maximum possible value of A.

3-4. Let G be a connected graph with $n > 1$ vertices. The maximal independent set of G is defined as the largest set of vertices so that no two are neighbours in G, and its size is denoted as $\alpha(G)$. Prove that there is an induced subgraph H of G with size at least $\alpha(G)/2$ where all degrees are odd.

3-5. (China 2022). Let m, n be two positive integers with $m \geq n \geq 2022$. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be $2n$ real numbers. Prove that the numbers of ordered pairs (i, j) $(1 \leq i, j \leq n)$ such that

$$|a_i + b_j - ij| \leq m$$

does not exceed $3n\sqrt{m \log n}$.

3-6. (RMM 2019). Given any positive real number ε, prove that, for all but finitely many positive integers v, any graph on v vertices with at least $(1+\varepsilon)v$ edges has two distinct simple cycles of equal lengths.

4 Problems

A1. (PUMaC 2019). A finite graph G is drawn on a blackboard. The following operation is permitted: pick any cycle of C of G, draw a new vertex v, connect it to all vertices of C, and finally erase all the edges of C. Prove that this operation can only be done a finite number of times.

A2. (Russia 2016). There are 30 teams in the NBA and every team plays 82 games in the year. The NBA Bosses want to partition the 30 teams into Western and Eastern Conferences (not necessary equally split), such that number of games between teams from different conferences is half the number of all games. Is it possible?

A3. (Russia 2000). Some pairs of cities in a certain country are connected by roads, at least three roads going out of each city. Prove that there exists a round path consisting of roads whose number is not divisible by 3.

A4. (Poland 1997). Given any n points on a unit circle show that at most $\frac{n^2}{3}$ of the segments joining two points have length $> \sqrt{2}$.

A5. (Russia 2003). There are N cities in a country. Any two of them are connected either by a road or by an airway. A tourist wants to visit every city exactly once and return to the city at which he started the trip. Prove that he can choose a starting city and make a path, changing means of transportation at most once.

A6. For which n is there a closed knight's tour on a $4 \times n$ chessboard?

A7. (RMM 2012). Given a finite number of boys and girls, a sociable set of boys is a set of boys such that every girl knows at least one boy in that set; and a sociable set of girls is a set of girls such that every boy knows at least one girl in that set. Prove that the number of sociable sets of boys and the number of sociable sets of girls have the same parity. (Acquaintance is assumed to be mutual.)

B1. (DMOPC 2018). Dr. Henri is trying to create a particular circuit. This circuit follows the edges of an $N \times 1$ grid of cells so that a resistor lies across each of the edges of the grid, except for the rightmost edge which is a power source (hence, there are $3N$ resistors.) Dr. Henri uses resistor chains. A resistor chain of length i is a chain of i resistors. He can bend a chain between its resistors, but he cannot split a chain. Dr. Henri has M resistor chains of lengths $a_1, a_2, \ldots, a_M$ and would like to use all of them in his circuit, so that there are no overlaps. Determine all M and a_i for which this is possible.

B2. (Taiwan 2023). Integers n and k satisfy $n > 2023k^3$. Kingdom Kitty has n cities, with at most one road between each pair of cities. It is known that the total number of roads in the kingdom is at least $2n^{3/2}$. Prove that we can choose $3k+1$ cities such that the total number of roads with both ends being a chosen city is at least $4k$.

B3. (Russia 1999). There are several cities in a country. Some pairs of the cities are connected by a two-way airline of one of the N companies, so that each company serves exactly one airline from each city, and one can travel between any two cities, possibly with transfers. During a financial crisis, $N-1$ airlines have been canceled, all from different companies. Prove that it is still possible to travel between any two cities.

B4. (CMO 2020). An undirected graph G has 19998 vertices. For any subgraph $\bar{G}$ of G with 9999 vertices, $\bar{G}$ has at least 9999 edges. Find the minimum number of edges in G.

B5. (USATST 2002). Let n be a positive integer and let S be a set of $2^n + 1$ elements. Let f be a function from the set of two-element subsets of S to $\{0, \ldots, 2^{n-1} - 1\}$. Assume that for any elements x, y, z of S, one of $f(\{x, y\}), f(\{y, z\}), f(\{z, x\})$ is equal to the sum of the other two. Show that there exist a, b, c in S such that $f(\{a, b\}), f(\{b, c\}), f(\{c, a\})$ are all equal to 0.

B6. (CMO 2019). A two-player game is played on $n \geq 3$ points, where no three points are collinear. Each move consists of selecting two of the points and drawing a new line segment connecting them. The first player to draw a line segment that creates an odd cycle loses. (An odd cycle must have all its vertices among the n points from the start, so the vertices of the cycle cannot be the intersections of the lines drawn.) Find all n such that the player to move first wins.

B7. (EGMO 2016). Let m be a positive integer. Consider a $4m \times 4m$ array of square unit cells. Two different cells are related to each other if they are in either the same row or in the same column. No cell is related to itself. Some cells are colored blue, such that every cell is related to at least two blue cells. Determine the minimum number of blue cells.

B8. (ISL 2012). The columns and the row of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is $0, 1$ or 2 respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color. Suppose that one can permute the tokens so that each token is moved to a distance of at most d from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 2$ from its original position, and each square contains a token with the same color as the square.

B9. (USAMO 2008). At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2^k for some positive integer k).

C1. (ISL 2018). Consider 2018 pairwise crossing circles no three of which are concurrent. These circles subdivide the plane into regions bounded by circular edges that meet at vertices. Notice that there are an even number of vertices on each circle. Given the circle, alternately colour the vertices on that circle red and blue. In doing so for each circle, every vertex is coloured twice- once for each of the two circle that cross at that point. If the two colours agree at a vertex, then it is assigned that colour; otherwise, it becomes yellow. Show that, if some circle contains at least 2061 yellow points, then the vertices of some region are all yellow.

C2. (CMO 2023). A country with n cities has some two-way roads connecting certain pairs of cities. Someone notices that if the country is split into two parts in any way, then there would be at most kn roads between the two parts (where k is a fixed positive integer). What is the largest integer m (in terms of n and k) such that there is guaranteed to be a set of m cities, no two of which are directly connected by a road?

C3. (IMO 2020). There are $4n$ pebbles of weights $1, 2, 3, \ldots, 4n$. Each pebble is coloured in one of n colours and there are four pebbles of each colour. Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:

- The total weights of both piles are the same.
- Each pile contains two pebbles of each colour.

C4. (Russia 2014). Two players play a card game. They have a deck of n distinct cards. For any pair of cards, one beats the other, possibly in a non-transitive fashion. The deck is split between players in an arbitrary manner. In each turn each player reveals the top card from their deck. The one whose card beats the other takes both and puts them to the bottom of their deck in any order of their discretion. Prove that for any initial distribution of cards, which the players are aware of, the players can agree and act so that one of the players is eventually left without a card.

C5. (Russia 1997). In an $m \times n$ rectangular grid, where m and n are odd integers, 1×2 dominoes are initially placed so as to exactly cover all but one of the 1×1 squares at one corner of the grid.

It is permitted to slide a domino towards the empty square, thus exposing another square.

Show that by a sequence of such moves, we can move the empty square to any corner of the rectangle.

C6. (IMO 2007). In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its size.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

C7. (ISL 2023). The Imomi archipelago consists of $n \geq 2$ islands. Between each pair of distinct islands is a unique ferry line that runs in both directions, and each ferry line is operated by one of k companies. It is known that if any one of the k companies closes all its ferry lines, then it becomes impossible for a traveller, no matter where the traveller starts at, to visit all the islands exactly once (in particular, not returning to the island the traveller started at).

Determine the maximal possible value of k in terms of n.

C8. (Japan 2004). There are a number of towns, each connected by roads with exactly other three towns. Last year we made a trip starting from a town and then returning to the town after visiting all other towns exactly once. This year we want to make another trip with this property. Prove that this is possible. Note that we don't want to use the exact same order (reverse order is also not allowed) as last year.

5 Hints

1-1. (i) Consider deleting edges iteratively until G is a tree. How should you choose these edges?

(ii) What if G has a cycle?

(iii) Induct on m.

1-2. Take the bipartite graph where an edge is formed between a row and column if their cell has a positive element. What can you say about each connected component?

1-3. It's easier to visualize as a Hamiltonian cycle on a grid. For the construction, induct on the dimension.

1-4. Use Hall's Marriage Theorem twice.

1-5. Build a directed graph corresponding to the 2^n strings and prove that it has an Eulerian tour.

1-6. Build a graph whose vertices are the residues mod n. For a piece from x to y, draw an edge between x to $y + 1$.

2-1. Consider the longest path.

2-2. What is the size of the largest clique?

2-3. Fix G and induct on the size of T

2-4. Delete vertices whose degree is less than half the average degree. What happens to the average degree?

2-5. What if G has an even cycle? What if G has no even cycles?

2-6. Neither operation increases the chromatic number of the graph.

3-1. Track the number of edges between those who have a ticket and those who don't.

3-2. Count the number of triples (U, V, W) for which $UV = VW = 1$.

3-3. Bound the number of triples (u, v, w) where (u, v) and (v, w) use the same language.

3-4. Split the maximal independent set into vertices of degree > 1 and $= 1$. Neighbours of the latter set can have their degree parities fixed. Use a probabilistic argument to get the desired bound.

3-5. When is a 4-cycle possible?

3-6. Start from a spanning forest and then add in the rest of the edges. Consider the lengths of the formed cycles.

A1. Compare the number of edges to the number of vertices.

A2. The graph has even degrees. What can you say about the number of edges between a partition?

A3. Consider the longest path.

A4. Use Turán's Theorem.

A5. Induct on N.

A6. The knight alternates between outer rows and inner rows.

A7. Define a formula for sociable sets based on edge indicator variables.

B1. Look at the odd degree vertices to bound M. The construction can be done by strong induction.

B2. Prove that there is a vertex contained by many 4-cycles.

B3. Say company N did not cancel any flights. Consider the union between company i's flights and company N's flights.

B4. For the subgraph H with 9999 vertices and minimal edges, prove that H and $G \setminus H$ have 3×9999 edges between them.

B5. Prove that there is a subset of $2^{n-1} + 1$ elements so that any two x, y in this subset have $f(\{x, y\})$ even.

B6. For n even, imagine pairing up the vertices across a mirror.

B7. Analyze connected components individually.

B8. Any token of colour A is matched to a token of colour A, B, and C, each of which is within 2 of a cell of colour A.

B9. Existence of at least one solution can be proved by induction.

C1. Show that any three circles either have two or six yellow points among their intersections.

C2. Consider when the graph is a disjoint union of cliques.

C3. The total weight condition can be guaranteed by pairing x and $4n+1-x$. Now build a graph where the vertices are the n colours.

C4. Each state of the game is a vertex that leads to two other states if no deck is empty. What is the in-degree of each vertex?

C5. Form the graph of squares which could be empty and prove that it is a tree.

C6. Begin with a maximal clique in one room and everyone else in the other. Move vertices from one room to another until the size of their maximal cliques differs by 1.

C7. Consider two vertices u, v whose edge is colour i. If $\deg_i(u) + \deg_i(v) \leq n - 1$, show that the edge u, v can be recoloured.

C8. Consider an edge (u, v) from the original Hamiltonian cycle. Removing it leaves two vertices of degree 2. The Hamiltonian path can be transformed into another path by using the edge from v to another neighbour. This defines an adjacency relationship between Hamiltonian paths.