Incircles

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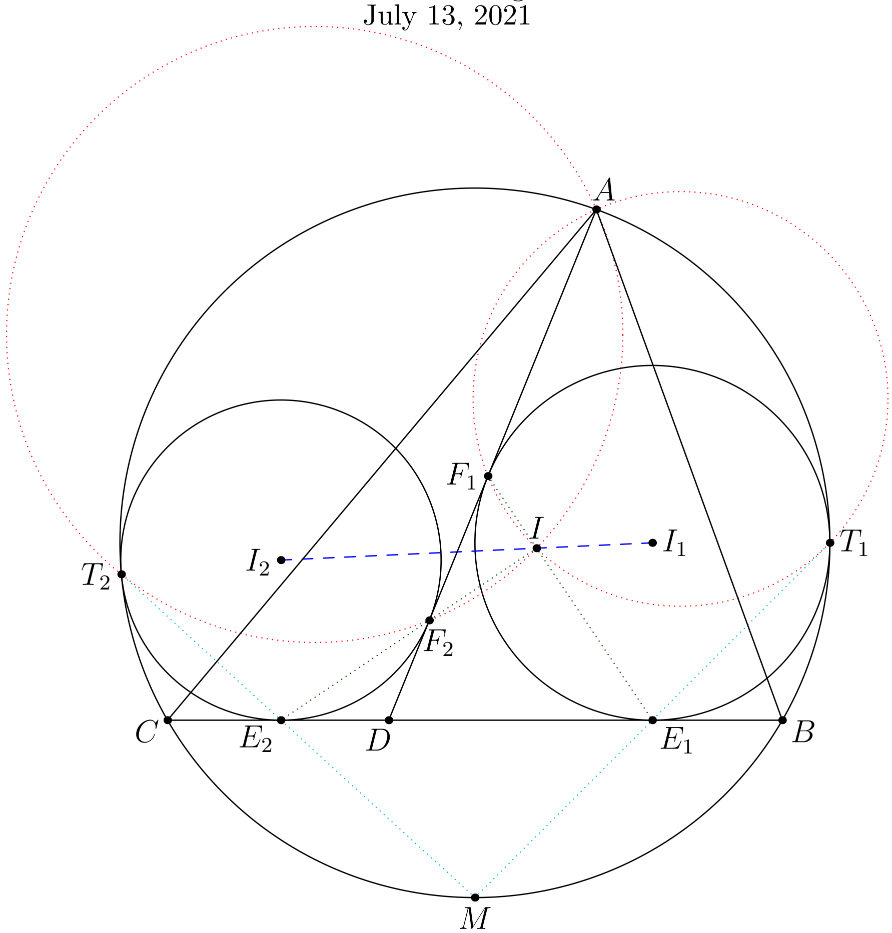


Lemma 1

Let AB be a chord in circle Ω and M be the midpoint of arc AB. Circle ω lies on the opposite side of AB as M and is tangent to segment AB and Ω at points P and Q respectively. Then P, Q, and M are collinear.

Lemma 2

Let Ω_1, Ω_2 be two circles intersecting at points A and B. Two circles ω_1, ω_2 lie on the same of AB and are each tangent to Ω_1 and Ω_2 at points P_1, Q_1 and P_2, Q_2, respectively. Then P_1, Q_1, P_2, and Q_2 are concyclic.

Theorem 1: Sawayama-Thébault

Let ABC be a triangle with incenter I. Let D be a point on BC. Let I_1 be the center of the circle that touches segments AD, DB, and the circumcircle of ABC, and let I_2 be the center of the circle that touches segments AD, DC, and the circumcircle of ABC. Then I_1, I_2, and I are collinear.

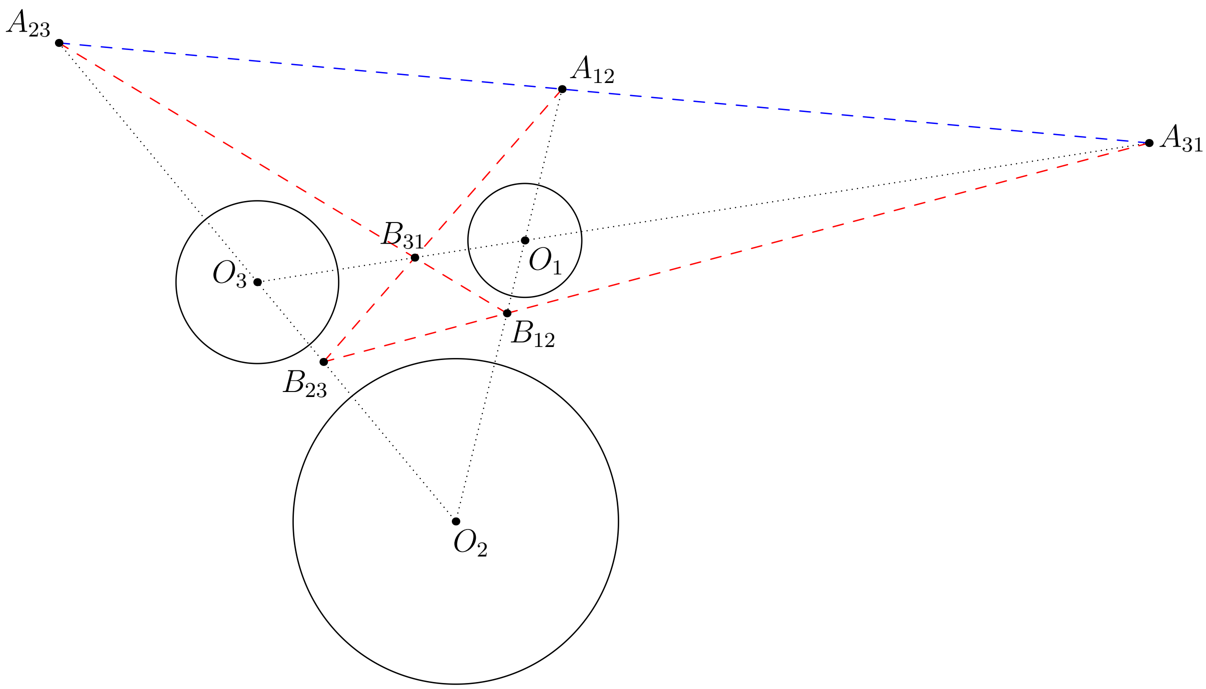
Theorem 2: Pitot's Theorem

Let $ABCD$ be a convex quadrilateral. Then $ABCD$ has an incircle if and only if

$$AB + CD = AD + BC.$$

Lemma 3

Let $ABCD$ be a bicentric quadrilateral with incenter I and circumcenter O. Let $WXYZ$ be the intouch quadrilateral. Then the diagonals of $ABCD$ and the diagonals of $WXYZ$ concur at a point P. Furthermore, P is on line OI.



Theorem 3: Monge's Theorem

Let $\omega_1, \omega_2, \omega_3$ be circles in the plane. Denote the external center of similitude between ω_j, ω_k as A_{jk} and the internal center of similitude as B_{jk}. Then for i, j, k a permutation of $\{1, 2, 3\}$, we have A_{ij}, A_{jk}, A_{ki} collinear and A_{ij}, B_{jk}, B_{ki} collinear.

A Problems

A1. Prove the lemmas and Monge's Theorem.

A2. Let $ABCD$ be a bicentric quadrilateral with intouch quadrilateral $WXYZ$. Prove that the diagonals of $WXYZ$ are perpendicular to one another.

A3. (ELMO 2011). Let $ABCD$ be a convex quadrilateral. Let E, F, G, H be points on segments AB, BC, CD, DA, respectively, and let P be the intersection of EG and FH. Given that quadrilaterals $HAEP$, $EBFP$, $FCGP$, $GDHP$ all have inscribed circles, prove that $ABCD$ also has an inscribed circle.

A4. (Japan MO 2009). Let Γ be a circumcircle. A circle with center O touches to line segment BC at P and touches the arc BC of Γ which doesn't have A at Q. If $\angle BAO = \angle CAO$, then prove that $\angle PAO = \angle QAO$.

A5. (ARMO 2018). Circle ω is tangent to sides AB, AC of triangle ABC. A circle Ω touches the side AC and line AB (produced beyond B), and touches ω at a point L on side BC. Line AL meets ω, Ω again at K, M, respectively. It turned out that $KB \parallel CM$. Prove that $\triangle LCM$ is isosceles.

A6. (ISL 2005). Given a triangle ABC satisfying $AC + BC = 3 \cdot AB$. The incircle of triangle ABC has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle.

A7. (CMO 2012). Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB$, $\angle ADB$ and $\angle APB$ meet at a common point.

B Problems

B1. (Germany MO 2009). Let $ABCD$ be a convex quadrilateral and let N be the intersection of diagonals AC and BD. Denote by a, b, c, d the length of the altitudes from N to AB, BC, CD, DA, respectively. Prove that $\frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d}$ if and only if $ABCD$ has an incircle.

B2. (USA TSTST 2017). Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

B3. (USA TST 2010). Let ABC be a triangle. Point M and N lie on sides AC and BC respectively such that $MN||AB$. Points P and Q lie on sides AB and CB respectively such that $PQ||AC$. The incircle of triangle CMN touches segment AC at E. The incircle of triangle BPQ touches segment AB at F. Line EN and AB meet at R, and lines FQ and AC meet at S. Given that $AE = AF$, prove that the incenter of triangle AEF lies on the incircle of triangle ARS.

B4. (ARMO 2016). In triangle ABC, $AB < AC$ and ω is the incircle. The A-excircle is tangent to BC at A'. Point X lies on AA' such that segment $A'X$ doesn't intersect with ω. The tangents from X to ω intersect with BC at Y, Z. Prove that the sum $XY + XZ$ not depends to point X.

B5. (ISL 2006). A point D is chosen on the side AC of a triangle ABC with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incenter of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.

B6. (ISL 2017). A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c and I_d be the incenters of the triangles DAB, ABC, BCD and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90^\circ$.

C Problems

C1. (ISL 2007). Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

C2. (ISL 2015). Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segment PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

C3. (China TST 2016). In cyclic quadrilateral $ABCD$, $AB > BC$, $AD > DC$, I, J are the incenters of $\triangle ABC$,$\triangle ADC$ respectively. The circle with diameter AC meets segment IB at X, and the extension of JD at Y. Prove that if the four points B, I, J, D are concyclic, then X, Y are the reflections of each other across AC.

C4. (IMO 2008). Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

C5. (Poland MO 2016). Let I be an incenter of $\triangle ABC$. Denote D, $S \neq A$ intersections of AI with BC, $O(ABC)$ respectively. Let K, L be incenters of $\triangle DSB$, $\triangle DCS$. Let P be a reflection of I with the respect to KL. Prove that $BP \perp CP$.

C6. (ISL 2009). Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2 and I_3 the incenters of $\triangle ABM$, $\triangle MNC$ and $\triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

C7. (ISL 2012). Let $ABCD$ be a convex quadrilateral with non-parallel sides BC and AD. Assume that there is a point E on the side BC such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point F on the side AD such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if AB is parallel to CD.

C8. (ISL 2010). Three circular arcs γ_1, γ_2, and γ_3 connect the points A and C. These arcs lie in the same half-plane defined by line AC in such a way that arc γ_2 lies between the arcs γ_1 and γ_3. Point B lies on the segment AC. Let h_1, h_2, and h_3 be three rays starting at B, lying in the same half-plane, h_2 being between h_1 and h_3. For $i, j = 1, 2, 3$, denote by V_{ij} the point of intersection of h_i and γ_j (see the Figure below). Denote by $\widehat{V_{ij}V_{kj}}\widehat{V_{kl}V_{il}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{il}$, $V_{kj}V_{kl}$ and arcs $V_{ij}V_{kj}$ and $V_{il}V_{kl}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs. Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}$, $\widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}$, $\widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed, too.