Complete Quadrilaterals and the Miquel Point

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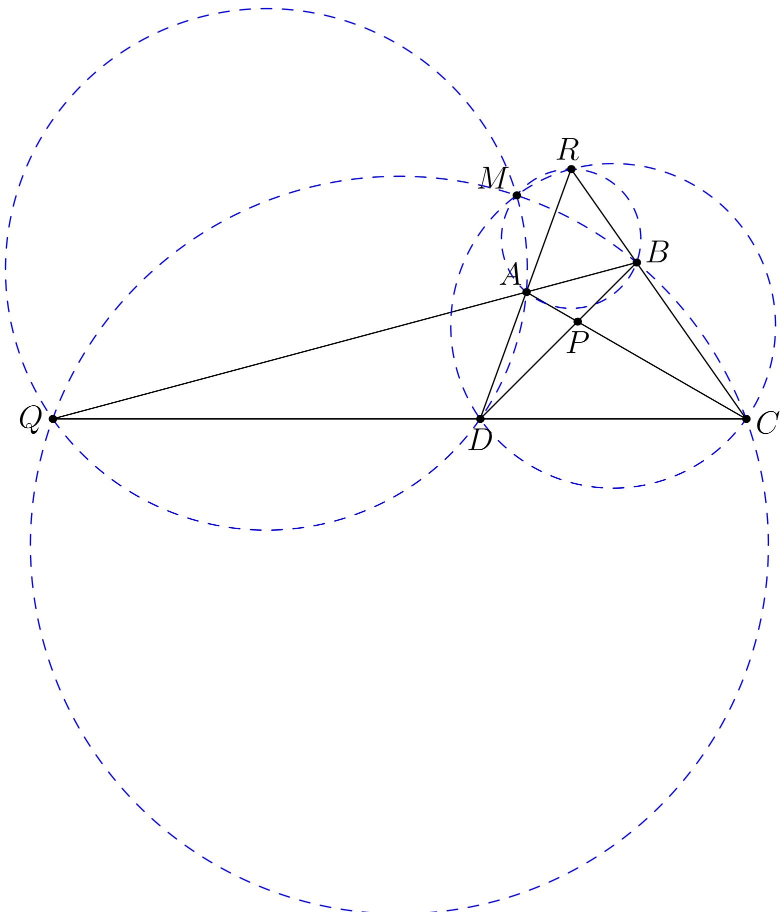
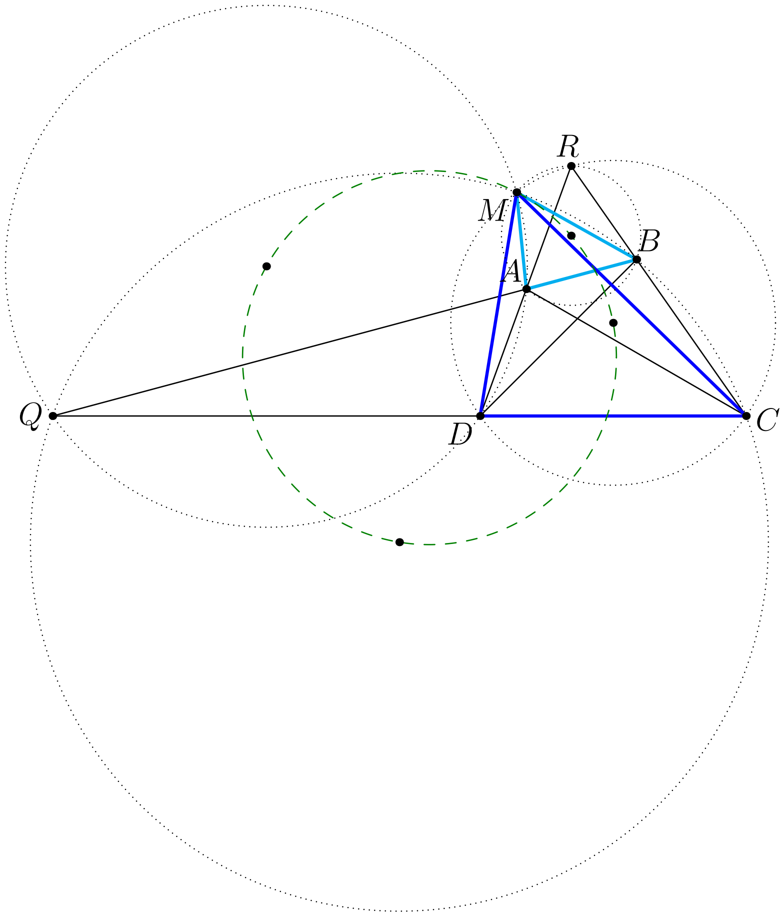


Theorem 1: Miquel's Theorem

Let ABC be a triangle and let D, E, F be points on lines BC, CA, and AB respectively. Then the circumcircles of AEF, BFD, and CDE intersect at a point known as the Miquel point.

Theorem 2: Miquel's Theorem for a Complete Quadrilateral

Let $ABCD$ be a quadrilateral and let $P := AC \cap BD$, $Q := AB \cap CD$, and $R := AD \cap BC$. Then the circumcircles of QAD, QBC, RAB, and RCD intersect at a point known as the Miquel point.



Lemma 1: Spiral Similarity

M is the center of spiral similarity for pairs of lines AB, CD; AD, BC; RA, CQ; RB, DQ; and RC, AQ.

Lemma 2

The angle bisectors for $\angle AMC$, $\angle BMD$, and $\angle QMR$ are the same line and

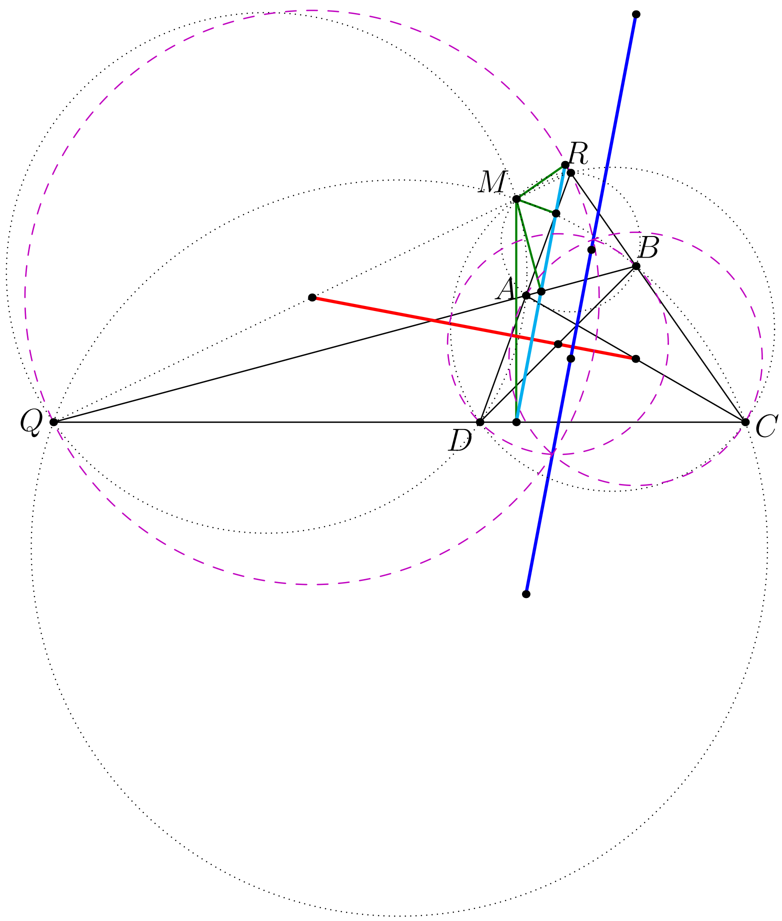
$$MA \cdot MC = MB \cdot MD = MQ \cdot MR.$$

Lemma 3

Inversion about M with power $MA \cdot MC$ and reflecting about the angle bisector of $\angle AMC$ overlays the diagram.

Lemma 4

M and the circumcenters of QAD, QBC, RAB, and RCD are concyclic.



Theorem 3: Simson Line of the Miquel Point

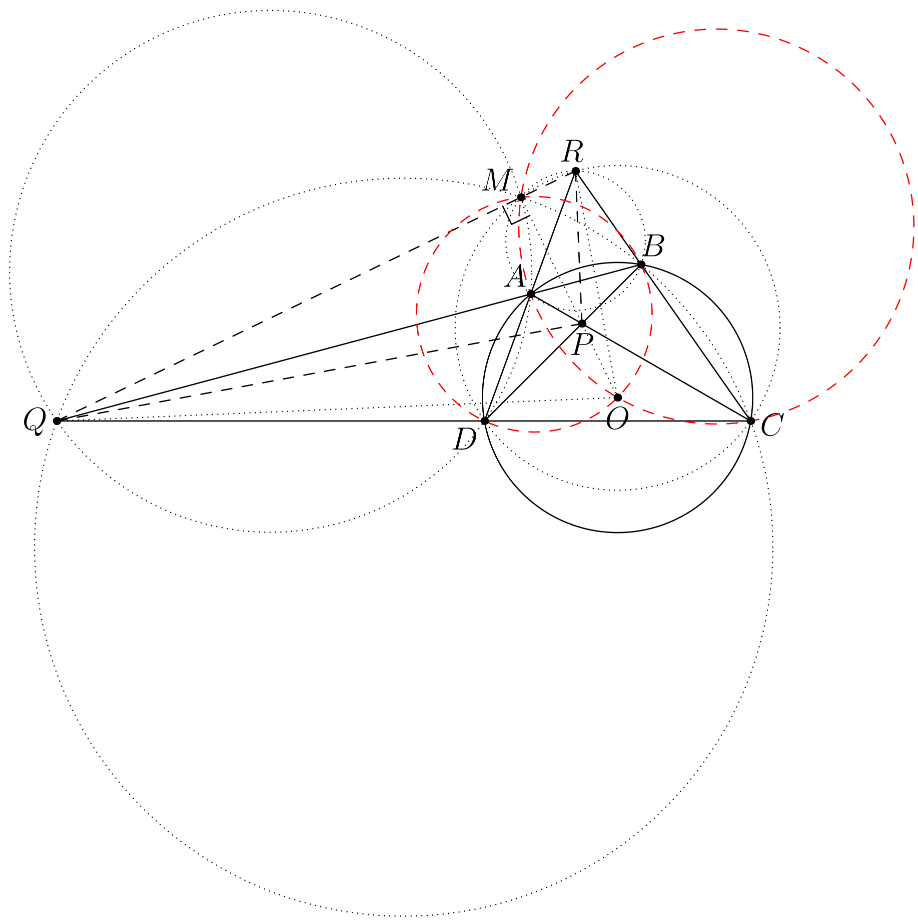
The feet of the perpendiculars of M to the four sides of the complete quadrilateral are collinear.

Theorem 4: Gauss-Bodenmiller Theorem

The circles with diameter AC, BD, and RQ are coaxial and the orthocenters of QAD, QBC, RAB, and RCD lie on this common axis.

Theorem 5: Newton-Gauss Line

The midpoints of the three diagonals (AC, BD, and QR) are collinear. Furthermore, this line is perpendicular to the Simson line of M.



Theorem 6: Brocard's Theorem

If $ABCD$ is cyclic with circumcenter O, then P is the pole of QR, Q is the pole of RP, R is the pole of PQ, and O is the orthocenter of triangle PQR.

Lemma 5

M lies on QR if and only if A, B, C, D are concyclic. Furthermore, if $ABCD$ is a cyclic quadrilateral, M is the foot of O on QR and P and M are inverses with respect to inversion about the circumcircle of $ABCD$.

Lemma 6

If $ABCD$ is cyclic with circumcenter O, M lies on the circumcircles of AOC and BOD. Furthermore, MO bisects $\angle AMC$ and $\angle BMD$.

A Problems

A1. Prove the theorems and lemmas. Some are tricky!

A2. (ToT 2015). Let $ABCD$ be a cyclic quadrilateral, K and N be the midpoints of the diagonals and P and Q be points of intersection of the extensions of the opposite sides. Prove that $\angle PKQ + \angle PNQ = 180$.

A3. (IMO 2013). Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

A4. (CGMO 2006). Let O be the intersection of the diagonals of convex quadrilateral $ABCD$. The circumcircles of $\triangle OAD$ and $\triangle OBC$ meet at O and M. Line OM meets the circumcircles of $\triangle OAB$ and $\triangle OCD$ at T and S respectively.

Prove that M is the midpoint of ST.

A5. (USAMO 2006). Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

A6. The Miquel point of a circumscribed quadrilateral $ABCD$ is M, and its incenter is I. Prove that the circumcircle of AMI is tangent to IC.

A7. (Serbia MO 2017). Let $ABCD$ be a convex and cyclic quadrilateral. Let $AD \cap BC = \{E\}$, and let M, N be points on AD, BC such that $AM/MD = BN/NC$. Circle around $\triangle EMN$ intersects circle around $ABCD$ at X, Y prove that AB, CD and XY are either parallel or concurrent.

A8. (USATST 2000). Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.

A9. (China TST 2008). Let ABC be a triangle and ℓ be a line which cuts lines BC, CA, and AB at D, E, and F, respectively. Denote by O_1, O_2, O_3 the circumcenters of triangles AEF, BFD, CDE, respectively. Prove that the orthocenter of triangle $O_1O_2O_3$ lies on ℓ.

B Problems

B1. (Serbia MO 2017). Let $ABCD$ be a convex cyclic quadrilateral. Let $E := AD \cap BC$ and let M, N be points on AD, BC, respectively, such that $AM : MD = BN : NC$. The circumcircle of $\triangle EMN$ intersects the circumcircle of $ABCD$ at points X and Y. Prove that AB, CD and XY are either parallel or concurrent.

B2. (China TST 2006). Let $ABCD$ be a convex cyclic quadrilateral with circumcenter O where O is not on any of the sides of $ABCD$. Let $P := AC \cap BD$. The circumcentres of $\triangle OAB$, $\triangle OBC$, $\triangle OCD$ and $\triangle ODA$ are O_1, O_2, O_3 and O_4 respectively.

Prove that O_1O_3, O_2O_4, and OP are concurrent.

B3. (ISL 2009). Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

B4. (Korea 2008). Quadrilateral $ABCD$ is inscribed in a circle with center O. Let $E := AB \cap CD$ and let P and R be the projections of E onto BC and AD, respectively. Let $Q := EP \cap AD, S := ER \cap BC$. Let K be the midpoint of QS. Prove that E, K, O are collinear.

B5. (CGMO 2019). Let $ABCD$ be a cyclic quadrilateral. Line AC intersects BD at P, and line BC intersects AD at Q. Let M be the midpoint of CD and let the reflection of line AB over line PQ intersect CD at K. Prove that P, Q, M, K are concyclic.

B6. (ARMO 2019). Let ABC be an acute-angled triangle with $AC < BC$. A circle passes through A and B and crosses the segments AC and BC again at A_1 and B_1 respectively. The circumcircles of A_1B_1C and ABC meet each other at points P and C. The segments AB_1 and A_1B intersect at S. Let Q and R be the reflections of S in the lines CA and CB respectively. Prove that the points P, Q, R, and C are concyclic.

B7. (Taiwan TST 2018). Given a $\triangle ABC$ with circumcircle Ω and a point P. Let D be the second intersection of AP with Ω, E, F be the intersection of BP, CP with CA, AB, respectively, M be the intersection of $\odot(AEF)$ with Ω, T be the intersection of the tangent of Ω at B, C and U be the second intersection of TD with Ω. Prove that the reflection of U in BC lies on $\odot(DMP)$.

B8. (USAMO 2018). In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90°$.

C Problems

C1. (China TST 2016). The diagonals of a cyclic quadrilateral $ABCD$ intersect at P, and there exist a circle Γ tangent to the extensions of AB, BC, AD, DC at X, Y, Z, T respectively. Circle Ω passes through points A, B, and is externally tangent to circle Γ at S. Prove that $SP \perp ST$.

C2. (Brazil MO 2016). Let $ABCD$ be a non-cyclic, convex quadrilateral, with no parallel sides. The lines AB and CD meet in E. Let $M \neq E$ be the intersection of circumcircles of ADE and BCE. The internal angle bisectors of $ABCD$ form a convex, cyclic quadrilateral with circumcenter I. The external angle bisectors of $ABCD$ form a convex, cyclic quadrilateral with circumcenter J. Show that I, J, M are collinear.

C3. (ISL 2006). Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

C4. (ISL 2012). Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

C5. (IMO 2011). Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively.

Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

C6. Prove that the Miquel point of a complete quadrilateral lies on the nine-point circle of the triangle determined by its three diagonals.

C7. Let I be the incenter of $\triangle ABC$. Let P be a point lie on the incircle (I) of $\triangle ABC$. Let ℓ be a line passing through P and tangent to (I). Let ℓ_a, ℓ_b, ℓ_c be the reflection of ℓ in BC, CA, AB, respectively. Let M be the Miquel point of complete quadrilateral $\{\ell, \ell_a, \ell_b, \ell_c\}$. Find the locus of M as P varies on (I).