Functional Equations in Number Theory

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A Problems

A1. (ISL 2013). Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

A2. (USAMO 2012). Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of positive integers) such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

A3. (APMO 2019). Let $\mathbb{Z}^+$ be the set of positive integers. Determine all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a, b.

A4. (ISL 2004). Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$\left(f^2(m) + f(n)\right) \mid \left(m^2 + n\right)^2$$

for any two positive integers m and n.

A5. (USAMO 2019). Let $\mathbb{N}$ be the set of positive integers. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies the equation

$$\underbrace{f(f(\ldots f}_{f(n)\text{ times}}(n)\ldots)) = \frac{n^2}{f(f(n))}$$

for all positive integers n. Given this information, determine all possible values of $f(1000)$.

A6. (USATST 2015). Let $f : \mathbb{Q} \to \mathbb{Q}$ be a function such that for any $x, y \in \mathbb{Q}$, the number $f(x + y) - f(x) - f(y)$ is an integer. Decide whether it follows that there exists a constant c such that $f(x) - cx$ is an integer for every rational number x.

B Problems

B1. (ISL 2011). Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself, such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

B2. (IMO 2011). Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m - n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.

B3. (ISL 2016). Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

B4. (USATST 2019). Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x) + x, \quad g(x) + 2x, \quad \dots, \quad g(x) + 100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

B5. (ISL 2008). For every $n \in \mathbb{N}$ let $d(n)$ denote the number of (positive) divisors of n. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the following properties:

- $d(f(x)) = x$ for all $x \in \mathbb{N}$.
- $f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all $x, y \in \mathbb{N}$.

B6. (ISL 2007). Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime p, the number $f(m+n)$ is divisible by p if and only if $f(m) + f(n)$ is divisible by p.

C Problems

C1. (IMO 1998). Determine the least possible value of $f(1998)$, where $f : \mathbb{N} \to \mathbb{N}$ is a function such that for all $m, n \in \mathbb{N}$,

$$f\left(n^2 f(m)\right) = m\left(f(n)\right)^2.$$

C2. (IMO 2010). Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$\left(g(m) + n\right)\left(g(n) + m\right)$$

is a perfect square for all $m, n \in \mathbb{N}$.

C3. (ISL 2018). Let $f : \{1, 2, 3, \dots\} \to \{2, 3, \dots\}$ be a function such that $f(m+n) | f(m) + f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f.

C4. (HMIC 2019). Let $p = 2017$ be a prime and $\mathbb{F}_p$ be the integers modulo p. A function $f : \mathbb{Z} \to \mathbb{F}_p$ is called good if there is $\alpha \in \mathbb{F}_p$ with $\alpha \not\equiv 0 \pmod{p}$ such that

$$f(x)f(y) = f(x+y) + \alpha^y f(x-y) \pmod{p}$$

for all $x, y \in \mathbb{Z}$. How many good functions are there that are periodic with minimal period 2016?

C5. (ISL 2015). Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer k, a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called k-good if $\gcd(f(m) + n, f(n) + m) \leq k$ for all $m \neq n$. Find all k such that there exists a k-good function.

C6. (ISL 2017). Let p be an odd prime number and $\mathbb{Z}_{>0}$ be the set of positive integers. Suppose that a function $f : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \{0, 1\}$ satisfies the following properties:

- $f(1, 1) = 0$.
- $f(a, b) + f(b, a) = 1$ for any pair of relatively prime positive integers (a, b) not both equal to 1;
- $f(a + b, b) = f(a, b)$ for any pair of relatively prime positive integers (a, b).

Prove that

$$\sum_{n=1}^{p-1} f(n^2, p) \geqslant \sqrt{2p} - 2.$$