Polynomials in Number Theory

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A Problems

A1. (ELMO 2016). Big Bird has a polynomial P with integer coefficients such that n divides $P(2^n)$ for every positive integer n. Prove that Big Bird's polynomial must be the zero polynomial.

A2. (Schur's Theorem). For any integer polynomial $P(x)$, prove that either $P(x)$ is constant or there are infinitely many primes which divide $P(n)$ for some integer n.

A3. (ELMOSL 2014). It is well-known that the 3-variable polynomial $a^3 + b^3 + c^3 - 3abc$ can be factored as $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$. Prove that for $n > 3$,

$$P(a_1, a_2, \ldots, a_n) := a_1^n + a_2^n + \cdots + a_n^n - na_1a_2 \cdots a_n$$

is irreducible over $\mathbb{Z}[a_1, a_2, \ldots, a_n]$.

Bonus: Can you show the same over $\mathbb{C}[a_1, a_2, \ldots, a_n]$?

A4. (CMO 2016). Find all polynomials $P(x)$ with integer coefficients such that $P(P(n) + n)$ is a prime number for infinitely many integers n.

A5. (CMO 2010). Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Let $a_n = n! + n$. Show that if $\frac{P(a_n)}{Q(a_n)}$ is an integer for every n, then $\frac{P(n)}{Q(n)}$ is an integer for every integer n such that $Q(n) \neq 0$.

A6. (USAMTS 2018). A nonnegative integer is called *uphill* if its decimal digits are non-decreasing from left to right (0 is considered to be uphill). A polynomial $P(n)$ has rational coefficients and $P(n)$ is an integer for every uphill number n. Is it necessarily true that $P(n)$ is an integer for all integers n?

A7. (ISL 2012). Consider a polynomial $P(x) = \prod_{j=1}^{9}(x + d_j)$, where $d_1, d_2, \ldots d_9$ are nine distinct integers. Prove that there exists an integer N, such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

B Problems

B1. (Folklore). If $a_1, a_2, \ldots, a_n$ are distinct integers, prove that the polynomial $P(x) = (x-a_1)(x-a_2)\cdots(x - a_n) - 1$ is irreducible over integer polynomials.

B2. (USATST 2012). Consider (3-variable) polynomials

$$P_n(x, y, z) = (x - y)^{2n}(y - z)^{2n} + (y - z)^{2n}(z - x)^{2n} + (z - x)^{2n}(x - y)^{2n}$$

and

$$Q_n(x, y, z) = [(x - y)^{2n} + (y - z)^{2n} + (z - x)^{2n}]^{2n}.$$

Determine all positive integers n such that the quotient $Q_n(x, y, z)/P_n(x, y, z)$ is a (3-variable) polynomial with rational coefficients.

B3. (ISL 2009). Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n-fold application of T.

B4. (ISL 2012). For a nonnegative integer n define $\operatorname{rad}(n) = 1$ if $n = 0$ or $n = 1$, and $\operatorname{rad}(n) = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are all prime factors of n. Find all polynomials $f(x)$ with nonnegative integer coefficients such that $\operatorname{rad}(f(n))$ divides $\operatorname{rad}(f(n^{\operatorname{rad}(n)}))$ for every nonnegative integer n.

B5. (ISL 2012). Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

B6. (IMO 2006). Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x)) \ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

B7. (APMO 2018). Find all polynomials $P(x)$ with integer coefficients such that for all real numbers s and t, if $P(s)$ and $P(t)$ are both integers, then $P(st)$ is also an integer.

C Problems

C1. (IMO 2002). Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

C2. (IMO 2017). An ordered pair (x, y) of integers is a primitive point if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, \ldots, a_n$ such that, for each (x, y) in S, we have:

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_{n-1} x y^{n-1} + a_n y^n = 1.$$

C3. (ISL 2011). Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients, such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)} - 1$ divides $3^{P(n)} - 1$. Prove that $Q(x)$ is a constant polynomial.

C4. (USATSTST 2016). Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \ldots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo n.

C5. (USATSTST 2019). Suppose P is a polynomial with integer coefficients such that for every positive integer n, the sum of the decimal digits of $|P(n)|$ is not a Fibonacci number. Must P be constant? (A Fibonacci number is an element of the sequence $F_0, F_1, \ldots$ defined recursively by $F_0 = 0, F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$.)