Projective Geometry

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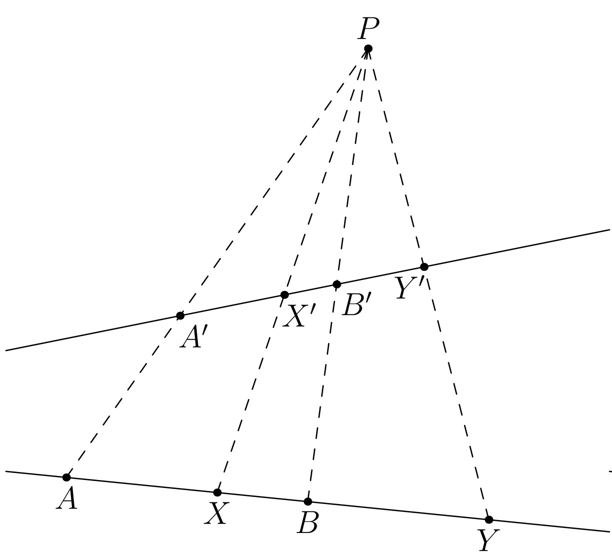
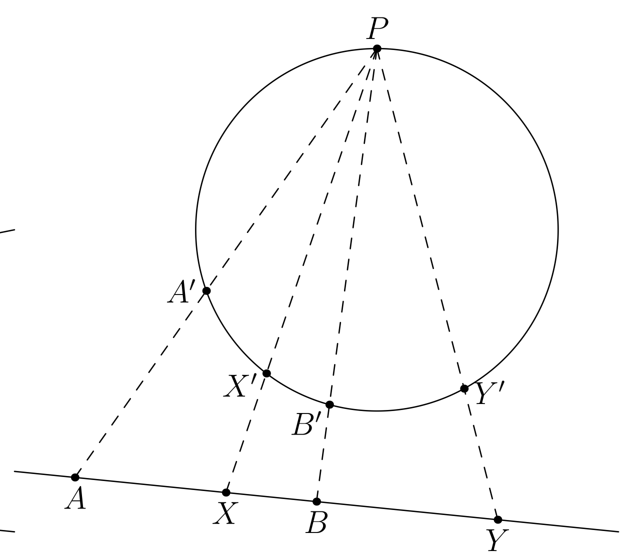
Introduction

Projective geometry deals with elements of geometry which are preserved under projections such as points, lines, intersections, tangencies, conics. There are a number of useful concepts and tools deriving from this area.

The **projective plane** extends the usual Euclidean plane to resolve parallel lines. While in Euclidean geometry, two distinct lines do not intersect if the are parallel and at one point otherwise, in the projective plane, two parallel lines meet at the **point at infinity**. This is often denoted as P_∞. Note that different families of parallel lines meet at different points at infinity and these points constitute the **line at infinity**. This extension allows us to handle special cases well.

Projective geometry revolves around intersections and taking perspectives is a common way in which we can relate groups of intersections.





The **cross-ratio** of four collinear points A, B, X, Y is denoted and defined as

$$(A, B; X, Y) := \frac{XA}{XB} \div \frac{YA}{YB}$$

where the lengths are directed. Furthermore, we can define the crossratio of four concurrent lines PA, PB, PX, PY as

$$P(A, B; X, Y) := \frac{\sin \angle XPA}{\sin \angle XPB} \div \frac{\sin \angle YPA}{\sin \angle YPB}.$$

Theorem 1: Cross-ratio Under Perspectivity

For any four collinear points A, B, X, Y and point P not on this line, we have

$$P(A, B; X, Y) = (A, B; X, Y).$$

As an immediate corollary, we have

$$(A, B; X, Y) \stackrel{P}{=} (A', B'; X', Y')$$

for A', B', X', Y' defined as in the left diagram.

We can also extend this notion of cross-ratios to four concyclic points. We can (in a slight abuse of notation) write $(A, B; X, Y) := \frac{XA}{XB} \div \frac{YA}{YB}$ for concyclic points A, B, X, Y. Then we have the following:

Theorem 2: Cross-ratio Under Perspectivity

For any four concyclic points A, B, X, Y and point P on this same circle, we have

$$P(A, B; X, Y) = (A, B; X, Y).$$

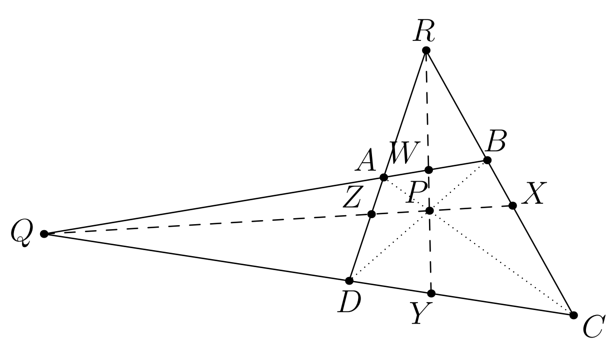
Again, we have that

$$(A, B; X, Y) \stackrel{P}{=} (A', B'; X', Y')$$

for A', B', X', Y' defined as in the right diagram. So cross-ratios are preserved under perspectivity.

Harmonic Stuff

Of particular interest are when the cross-ratio is equal to -1. In this case, we call the four points a **harmonic bundle**. Note that if $(A, B; X, Y) = -1$, then $(A, B; Y, X) = -1$) and given A, B, and X, we can uniquely define Y to complete the harmonic bundle. For example, let A and B be points with M their midpoint. Then $(A, B; M, P_\infty)$ is a harmonic bundle. Harmonic bundles arise naturally from cevians and complete quadrilaterals.



Lemma 1: Complete Quadrilaterals Induce Harmonic Bundles

There are a lot of harmonic bundles in the above diagram.

One particularly useful harmonic bundle is the feet of the angle bisectors.

Lemma 2: Feet of Angle Bisectors Form Harmonic Bundles

et X, A, Y, and B be collinear points in that order and let C be any point not on this line. Then any two of the following conditions imply the third:

(i) $(A, B; X, Y)$ is a harmonic bundle.

(ii) $\angle XCY = 90°$.

(iii) CY bisects $\angle ACB$.

Finally, if A, B, X, Y is cyclic and $(A, B; X, Y) = -1$, then we say that $AXBY$ is a **harmonic quadrilateral**. These are closely related to the symmedian configuration, as the next theorem shows.

Lemma 3: Symmedians Form Harmonic Quadrilaterals

Let ABC be a triangle and Γ its circumcircle. Let D be the point on arc BC of Γ not containing A such that $ABDC$ is harmonic. Let M be the midpoint of BC and let the tangents to Γ at B and C intersect at X. Then

(i) AM and AD are isogonal lines with respect to $\angle BAC$.

(ii) D is the second intersection of AX with Γ.

Poles and Polars

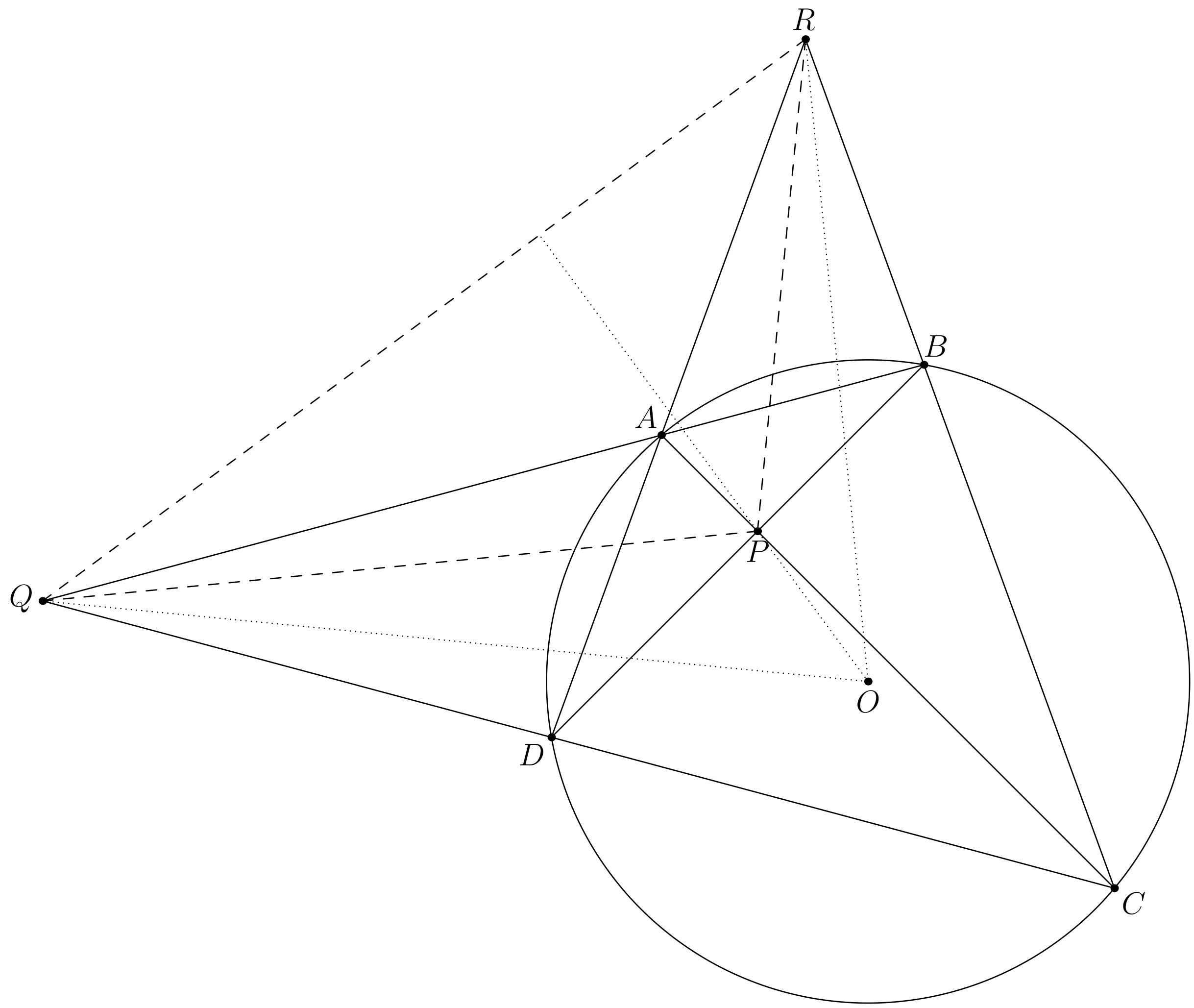
Let Ω be a circle with center O and let P be a point. A line is drawn through P and intersects Ω twice at points A and B. Point Q is chosen to be the point such that P, Q, A, B is a harmonic bundle. Then as the line through P varies, the Q moves along a locus ℓ known as the **polar of** P (and conversely, P is the **pole of** ℓ).

Theorem 3: Pole and Polar

The polar of P, ℓ, is a line perpendicular to OP which goes through the inverse of P with respect to Ω.

Theorem 4: La Hire's Theorem

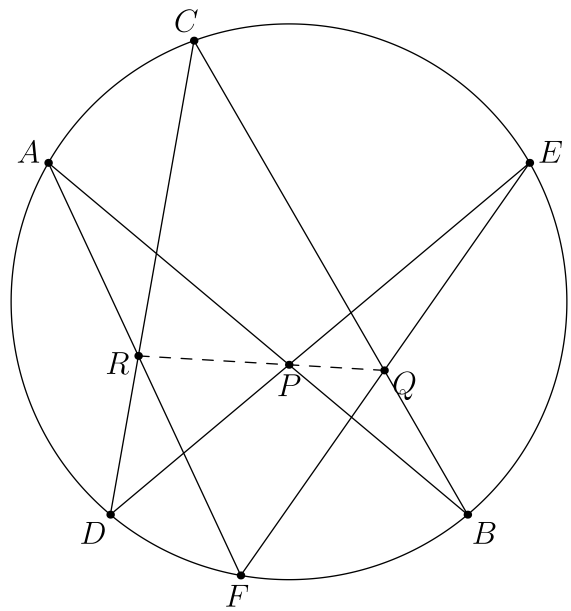
A point X lies on the polar of Y if and only if Y lies on the polar of X.



Theorem 5: Brocard's Theorem

Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let $P := AC \cap BD$, $Q := AB \cap CD$, and $R := BC \cap DA$. Then P is the pole of QR, Q is the pole of RP, R is the pole of PQ, and O is the orthocenter of triangle PQR.

Pascal, Pappus, and Brianchon, Oh My!



Theorem 6: Pascal's Theorem

Let points A, B, C, D, E, F lie on a circle. Let $P := AB \cap DE, Q := BC \cap EF, R := CD \cap FA$. Then P, Q, R are collinear.

Theorem 7: Pappus's Theorem

Let points A, B, C, D, E, F such that A, C, E are collinear and B, D, F are collinear. Let $P := AB \cap DE, Q := BC \cap EF, R := CD \cap FA$. Then P, Q, R are collinear.

Theorem 8: Brianchon's Theorem

Let $ABCDEF$ be a hexagon circumscribed about a circle. Then AD, BE, and CF concur.

A Problems

A1. Prove all the lemmas and theorems.

A2. Let ABC be a triangle and let D, E, and F be points on sides BC, CA, and AB respectively. Prove that $BD = DC$ if and only if $EF||BC$.

A3. Let ABC be a triangle with incenter I and A-excenter I_A. Let the internal angle bisector of $\angle BAC$ intersect BC at D. Prove that A, D, I, I_A forms a harmonic bundle.

A4. (CMO 1994). Let ABC be an acute triangle. Let D be on BC with $AD \perp BC$ and let H be any point on segment AD. Lines BH and CH, when extended, intersect AC and AB at E and F respectively. Prove that $\angle EDH = \angle FDH$.

A5. Points A, X, B, Y lie on a line in that order and $(A, B; X, Y) = -1$. Let M be the midpoint of AB. Show that $MX \cdot MY = \left(\frac{1}{2}AB\right)^2$ and $YX \cdot YM = YA \cdot YB$.

A6. (USAJMO 2011). Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $DE \parallel AC$. Prove that BE bisects AC.

A7. (USAJMO 2015). Let $ABCD$ be a cyclic quadrilateral. Prove that there exists a point X on segment $\overline{BD}$ such that $\angle BAC = \angle XAD$ and $\angle BCA = \angle XCD$ if and only if there exists a point Y on segment $\overline{AC}$ such that $\angle CBD = \angle YBA$ and $\angle CDB = \angle YDA$.

A8. (Iran MO 2015). In quadrilateral $ABCD$, AC is the bisector of $\angle A$ and $\angle ADC = \angle ACB$. Let X and Y be the feet of perpendiculars from A to BC and CD, respectively. Prove that the orthocenter of triangle AXY is on BD.

B Problems

B1. (Taiwan TST 2015). Let O be the circumcircle of triangle ABC. Two circles O_1, O_2 are tangent to each of the circle O and the rays $\overrightarrow{AB}, \overrightarrow{AC}$, with O_1 interior to O, O_2 exterior to O. The common tangent of O, O_1 and the common tangent of O, O_2 intersect at the point X. Let M be the midpoint of the arc BC (not containing the point A) on the circle O, and the segment $\overline{AA'}$ be the diameter of O. Prove that X, M, and A' are collinear.

B2. (Iran TST 2012). Let ABC be an acute triangle with circumcircle ω. Let D be the midpoint of arc BAC and I be the incenter of triangle ABC. Let DI intersect BC in E and ω for second time in F. Let P be a point on line AF such that PE is parallel to AI. Prove that PE bisects $\angle BPC$.

B3. (APMO 2008). Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides BC and BA at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines of Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.

B4. (Iran TST 2008). Suppose that I is incenter of triangle ABC and l' is a line tangent to the incircle. Let l be another line such that intersects AB, AC, BC respectively at C', B', A'. We draw a tangent from A' to the incircle other than BC, and this line intersects with l' at A_1. B_1, C_1 are similarly defined. Prove that AA_1, BB_1, CC_1 are concurrent.

B5. (APMO 2013). Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

B6. (IMO 2010). Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

B7. (Iran MO 2017). Let ABC be an acute-angle triangle. Let M be the midpoint of BC and H be the orthocenter of ABC. Let $F := BH \cap AC$ and $E := CH \cap AB$. Suppose that X is a point on EF such that $\angle XMH = \angle HAM$ and A, X are in the distinct side of MH. Prove that AH bisects MX.

B8. (Taiwan TST 2020). Let point H be the orthocenter of a scalene triangle ABC. Line AH intersects with the circumcircle Ω of triangle ABC again at point P. Line BH, CH meets with AC, AB at point E and F, respectively. Let PE, PF meet Ω again at point Q, R, respectively. Point Y lies on Ω so that lines AY, QR and EF are concurrent. Prove that PY bisects EF.

B9. (USA TST 2013). Let ABC be an acute triangle. Circle ω_1, with diameter AC, intersects side BC at F (other than C). Circle ω_2, with diameter BC, intersects side AC at E (other than C). Ray AF intersects ω_2 at K and M with $AK < AM$. Ray BE intersects ω_1 at L and N with $BL < BN$. Prove that lines AB, ML, NK are concurrent.

B10. (ISL 2016). Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

B11. (USA TSTST 2020). Let ABC be a scalene triangle with incenter I. The incircle of ABC touches $\overline{BC}, \overline{CA}, \overline{AB}$ at points D, E, F, respectively. Let P be the foot of the altitude from D to $\overline{EF}$, and let M be the midpoint of $\overline{BC}$. The rays AP and IP intersect the circumcircle of triangle ABC again at points G and Q, respectively. Show that the incenter of triangle GQM coincides with D.

C Problems

C1. (Iran TST 2009). In triangle ABC, D, E and F are the points of tangency of incircle with the center of I to BC, CA and AB respectively. Let M be the foot of the perpendicular from D to EF. P is on DM such that $DP = MP$. If H is the orthocenter of BIC, prove that PH bisects EF.

C2. (USA TST 2013). Let ABC be a scalene triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. The circumcircle of triangle DKL intersects segment AB at a second point T (other than D). Prove that $\angle ACT = \angle BCT$.

C3. (ISL 2016). Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments

BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

C4. (USA TST 2017). Let ABC be a triangle with altitude $\overline{AE}$. The A-excircle touches $\overline{BC}$ at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.

C5. (ISL 2019). Let I be the incentre of acute-angled triangle ABC. Let the incircle meet BC, CA, and AB at D, E, and F, respectively. Let line EF intersect the circumcircle of the triangle at P and Q, such that F lies between E and P. Prove that $\angle DPA + \angle AQD = \angle QIP$.

C6. (IMO 2019). Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

C7. (Iran TST 2020). Given a triangle ABC with circumcircle Γ. Points E and F are the foot of angle bisectors of B and C, I is incenter and K is the intersection of AI and EF. Suppose that T be the midpoint of arc BAC. Circle Γ intersects the A-median and circumcircle of AEF for the second time at X and S. Let S' be the reflection of S across AI and J be the second intersection of circumcircle of $AS'K$ and AX. Prove that quadrilateral $TJIX$ is cyclic.