Ratio Chasing

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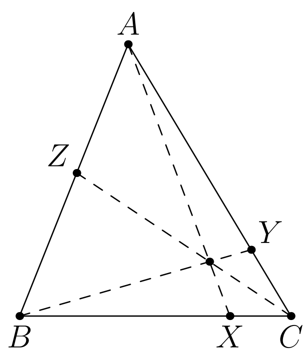
1 Concurrency and Collinearity

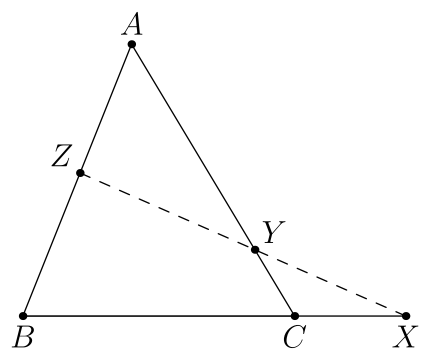
In a triangle $\triangle ABC$, a *cevian* is a line segment connecting one of the vertices A, B, or C to a point lying on the side opposite to the vertex. Ceva's Theorem relates the concurrency of three such cevians to length ratios.

Theorem 1: Ceva's Theorem

Given $\triangle ABC$, let X, Y, and Z be points on lines BC, CA, and AB respectively. Then AX, BY, and CZ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$



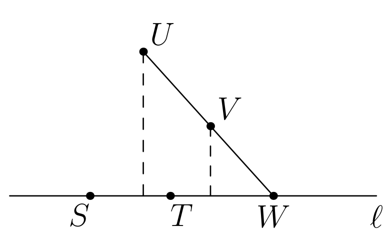


Before proving this theorem, let us consider the following lemma:

Lemma 1

Let ℓ be a line and U, V points in the plane. Let S and T be points on ℓ. Line UV intersects ℓ at point W. Then

$$\frac{[\triangle UST]}{[\triangle VST]} = \frac{d(U,\ell)}{d(V,\ell)} = \frac{UW}{VW}.$$



Proof. We will first show that if AX, BY, and CZ concur, the given equation is true. Let P be the point of intersection of the three lines. From Lemma 1, $\frac{[\triangle PAB]}{[\triangle PCA]} = \frac{BX}{XC}$. Similarly, $\frac{[\triangle PBC]}{[\triangle PAB]} = \frac{CY}{YA}$ and $\frac{[\triangle PCA]}{[\triangle PBC]} = \frac{AZ}{ZB}$. Multiplying the three inequalities gives

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{[\triangle PAB]}{[\triangle PCA]} \cdot \frac{[\triangle PBC]}{[\triangle PAB]} \cdot \frac{[\triangle PCA]}{[\triangle PBC]} = 1.$$

To prove the other direction, let P be the intersection of BY and CZ. Instead of directly proving that P lies on AX, we will define X' to be the intersection of AP and BC and show that $X = X'$. From the direction we have shown,

$$\frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

But we also have the assumption that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Dividing the two equations gives

$$\frac{BX'}{X'C} = \frac{BX}{XC}.$$

It is easy to see that this implies $X = X'$ and we are done. □

Theorem 2: Menelaus's Theorem

Given $\triangle ABC$, let X, Y, and Z be points on lines BC, CA, and AB respectively. Then X, Y, and Z are collinear if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$

The two theorems are almost the same save for the -1 in the right-hand side of Menelaus's Theorem and collinearity rather than concurrency. The negative sign arises from directed lengths. It is generally unnecessary to pay much attention when using directed lengths; they tend to work out without much trouble.

2 Angles

The Law of Sines provides a way of moving between ratios involving lengths and ratios involving angles.

Theorem 3: Extended Law of Sines

Let $\triangle ABC$ be a triangle with circumradius R. Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Theorem 4: Ratio Lemma

Given $\triangle ABC$, let X be a point on BC. Then

$$\frac{BX}{XC} = \frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{AB}{AC}.$$

Proof. Applying the Law of Sines to $\triangle ABX$ and $\triangle ACX$, we get

$$\frac{\sin \angle BAX}{BX} = \frac{\sin \angle BXA}{AB}$$

and

$$\frac{\sin \angle XAC}{XC} = \frac{\sin \angle CXA}{AC}.$$

Dividing these two equations and rearranging gives

$$\frac{\sin \angle BXA}{\sin \angle CXA} \cdot \frac{BX}{XC} = \frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{AB}{AC}.$$

Since $\angle BXA$ and $\angle CXA$ are supplementary, $\sin \angle BXA = \sin \angle CXA$. So we have

$$\frac{BX}{XC} = \frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{AB}{AC}$$

as expected. □

We can apply the Ratio Lemma to Ceva's Theorem and Menelaus's Theorem to obtain new formulations:

Theorem 5: Trigonometric Form of Ceva's Theorem

Given $\triangle ABC$, let X, Y, and Z be points on lines BC, CA, and AB respectively. Then AX, BY, and CZ are concurrent if and only if

$$\frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{\sin \angle CBY}{\sin \angle YBA} \cdot \frac{\sin \angle ACZ}{\sin \angle ZCB} = 1.$$

Theorem 6: Trigonometric Form of Menelaus's Theorem

Given $\triangle ABC$, let X, Y, and Z be points on lines BC, CA, and AB respectively. Then X, Y, and Z are collinear if and only if

$$\frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{\sin \angle CBY}{\sin \angle YBA} \cdot \frac{\sin \angle ACZ}{\sin \angle ZCB} = -1.$$

When working with these forms, we often obtain results about the ratio of sines of angles. The following lemma allows these results to cleanly translate into results about the angles themselves.

Lemma 2

Let x_1, y_1, x_2, y_2 be real numbers in $[0, \pi)$ such that $x_1 + y_1 = x_2 + y_2$, $x_1 + y_1 < \pi$ and $\frac{\sin x_1}{\sin y_1} = \frac{\sin x_2}{\sin y_2}$. Then $x_1 = x_2, y_1 = y_2$.

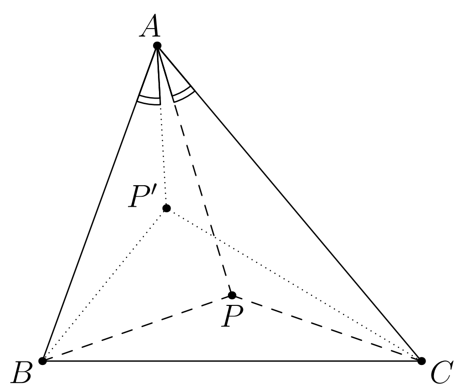
Note the bounds on $x_1 + y_1$; the lemma is not true when the angles sum to π and needs alterations when the angles sum to greater than π.

3 Isogonal Lines

Consider a point P and rays PU and PV. Two lines ℓ_1 and ℓ_2 passing through P are called *isogonal* lines if ℓ_1 and ℓ_2 are reflections over the bisector of $\angle UPV$. In other words, $\angle(\ell_1, PU) = \angle(\ell_2, PV)$ and $\angle(\ell_1, PV) = \angle(\ell_2, PU)$ (the two are equivalent).

Lemma 3: Existence of Isogonal Conjugate

Let $\triangle ABC$ be a triangle and P be some point. Then the isogonal of AP with respect to $\angle CAB$, the isogonal of BP with respect to $\angle ABC$, and the isogonal of CP with respect to $\angle BCA$ intersect at a point. This point is called the *isogonal conjugate* of P.



As an aside, the isogonal conjugate is what you get by flipping all the angles in trigonometric Ceva's. If you instead flipped all the side lengths, that is what's called the *isotomic conjugate*.

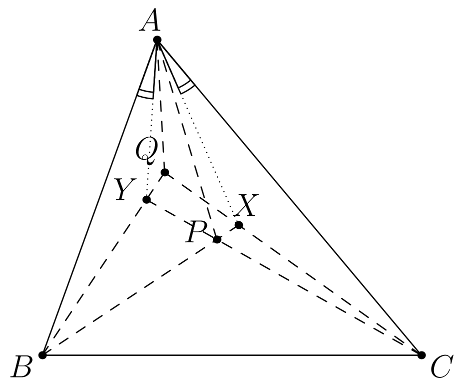
Lemma 4

Let $\triangle ABC$ be a triangle with points D and E on BC. Prove that AD and AE are isogonal with respect to $\angle CAB$ if and only if

$$\frac{BD}{DC} \cdot \frac{BE}{EC} = \left(\frac{AB}{AC}\right)^2.$$

Lemma 5: Isogonal Line Lemma

Given triangle ABC, let P and Q be points such that lines AP and AQ are isogonal lines with respect to $\angle BAC$. Define X as the intersection of lines BP and CQ and Y as the intersection of lines BQ and CP. Then AX and AY are isogonal lines with respect to $\angle BAC$.



Proof. We first apply the Ratio Lemma four times: on $\triangle AYQ$ with point B, on $\triangle AYP$ with point C, on $\triangle AXQ$ with point C, and on $\triangle AXP$ with point B. These give

$$\frac{AY}{AQ} \cdot \frac{\sin \angle YAB}{\sin \angle BAQ} = \frac{YB}{BQ} = \frac{[YBC]}{[QBC]},$$

$$\frac{AY}{AP} \cdot \frac{\sin \angle YAC}{\sin \angle CAP} = \frac{YC}{CP} = \frac{[YBC]}{[PBC]},$$

$$\frac{AX}{AQ} \cdot \frac{\sin \angle XAC}{\sin \angle CAQ} = \frac{XC}{CQ} = \frac{[XBC]}{[QBC]},$$

$$\frac{AX}{AP} \cdot \frac{\sin \angle XAB}{\sin \angle BAP} = \frac{XB}{BP} = \frac{[XBC]}{[PBC]}.$$

Now consider the product of the first and fourth equation, and the product of the second and third equations. These share many terms and we can divide the two, leaving

$$\frac{\sin \angle YAB}{\sin \angle BAQ} \cdot \frac{\sin \angle XAB}{\sin \angle BAP} = \frac{\sin \angle YAC}{\sin \angle CAP} \cdot \frac{\sin \angle XAC}{\sin \angle CAQ}.$$

We can now use the isogonality condition on AP and AQ to cancel out the denominators. Rearranging, we have

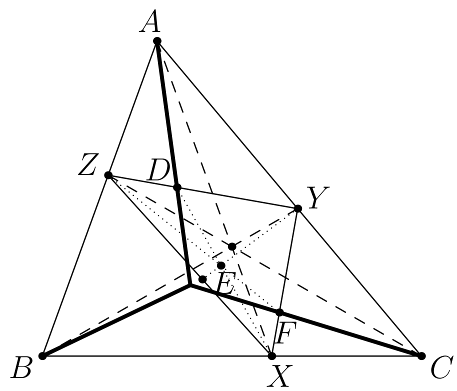
$$\frac{\sin \angle YAB}{\sin \angle YAC} = \frac{\sin \angle XAC}{\sin \angle XAB}.$$

By Lemma 2, this implies that $\angle YAB = \angle XAC$ and we are done. $\square$

4 Examples

Example 1: Cevian Nest Theorem

Let AX, BY, and CZ be concurrent cevians of $\triangle ABC$. Let XD, YE, and ZF be concurrent cevians in $\triangle XYZ$. Prove that AD, BE, and CF are concurrent.



Proof. By the Ratio Lemma on $\triangle AYZ$ with cevian AD,

$$\frac{YD}{DZ} = \frac{\sin\angle YAD}{\sin\angle DAZ} \cdot \frac{AY}{AZ}.$$

Similarly,

$$\frac{ZE}{EX} = \frac{\sin\angle ZBE}{\sin\angle EBX} \cdot \frac{BZ}{BX} \text{ and } \frac{XF}{FY} = \frac{\sin\angle XCF}{\sin\angle FCY} \cdot \frac{CX}{CY}.$$

Multiplying the three equations yields

$$\left(\frac{YD}{DZ} \cdot \frac{ZE}{EX} \cdot \frac{XF}{FY}\right) = \left(\frac{\sin\angle YAD}{\sin\angle DAZ} \cdot \frac{\sin\angle ZBE}{\sin\angle EBX} \cdot \frac{\sin\angle XCF}{\sin\angle FCY}\right) \cdot \left(\frac{AY}{AZ} \cdot \frac{BZ}{BX} \cdot \frac{CX}{CY}\right)$$

But from Ceva's Theorem on $\triangle XYZ$ with points D, E, F and on $\triangle ABC$ with points X, Y, Z, we know that

$$\frac{YD}{DZ} \cdot \frac{ZE}{EX} \cdot \frac{XF}{FY} = 1 \text{ and } \frac{AY}{AZ} \cdot \frac{BZ}{BX} \cdot \frac{CX}{CY} = \left(\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB}\right)^{-1} = 1.$$

Thus,

$$\frac{\sin\angle CAD}{\sin\angle DAB} \cdot \frac{\sin\angle ABE}{\sin\angle EBC} \cdot \frac{\sin\angle BCF}{\sin\angle FCA} = \frac{\sin\angle YAD}{\sin\angle DAZ} \cdot \frac{\sin\angle ZBE}{\sin\angle EBX} \cdot \frac{\sin\angle XCF}{\sin\angle FCY} = 1$$

and we are done. $\square$

Example 2

Let $\triangle ABC$ be a triangle with incenter I and contact triangle $\triangle DEF$. If M is the midpoint of BC, prove that EF, AM and DI concur.

Proof. Let X_1 be the intersection of AM and EF and let X_2 be the intersection of DI and EF. We wish to prove that $X_1 = X_2$. It suffices to show $\frac{EX_1}{X_1F} = \frac{EX_2}{X_2F}$.

Let us first compute $\frac{EX_1}{X_1F}$. From the Ratio Lemma on $\triangle AEF$ and on $\triangle ACB$ with cevian AX_1 and AM respectively, we obtain

$$\frac{EX_1}{X_1F} = \frac{\sin\angle EAX_1}{\sin\angle X_1AF}\cdot\frac{AE}{AF} \text{ and } \frac{CM}{MB} = \frac{\sin\angle CAM}{\sin\angle MAB}\cdot\frac{AC}{AB}.$$

Dividing the two equations and using $\angle EAX_1 = \angle CAM, \angle X_1AF = \angle MAB$ gives

$$\begin{aligned}\frac{EX_1}{X_1F}\cdot\frac{MB}{CM} &= \frac{AE}{AF}\cdot\frac{AB}{AC}\\ \frac{EX_1}{X_1F} &= \frac{AB}{AC}.\end{aligned}$$

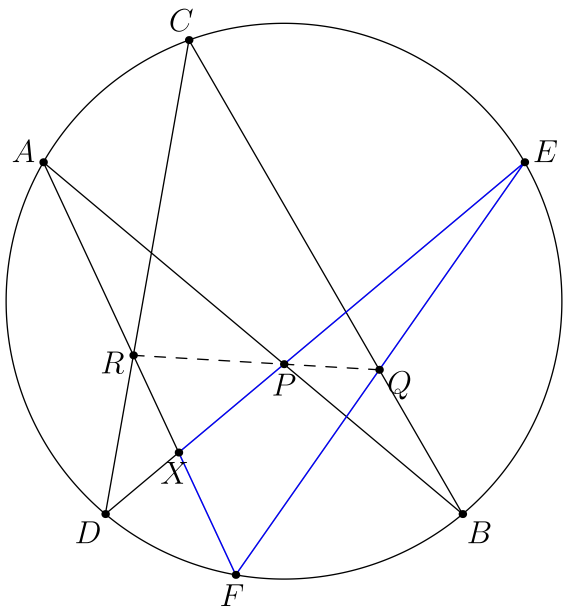
To compute $\frac{EX_2}{X_2F}$, we will apply the Ratio Lemma to $\triangle IEF$ with cevian IX_2. From angle chasing, we can see that $\angle EIX_2 = \angle BCA$ and $\angle X_2IF = \angle ABC$.

$$\begin{aligned}\frac{EX_2}{X_2F} &= \frac{\sin\angle EIX_2}{\sin\angle X_2IF}\cdot\frac{IE}{IF}\\ &= \frac{\sin\angle EIX_2}{\sin\angle X_2IF}\\ &= \frac{\sin\angle BCA}{\sin\angle ABC}\\ &= \frac{AB}{AC}.\end{aligned}$$

Thus, $X_1 = X_2$ and EF, AM, and DI concur. $\square$

Example 3: Pascal's Theorem

Let points A, B, C, D, E, F lie on a circle. Let $P := AB \cap DE, Q := BC \cap EF, R := CD \cap FA$. Prove that P, Q, R are collinear.



Proof. Define X to be the intersection of DE and AF. By Menelaus's Theorem on $\triangle EFX$, it suffices to show

$$\frac{EQ}{QF} \cdot \frac{FR}{RX} \cdot \frac{XP}{PE} = -1.$$

We can compute $\frac{EQ}{QF}$ by using the Ratio Lemma on $\triangle CEF$. In particular, we get

$$\frac{EQ}{QF} = \frac{CE}{CF} \cdot \frac{\sin \angle CEB}{\sin \angle BCF} = \frac{CE}{CF} \cdot \frac{EB}{BF}.$$

Similarly, by Ratio Lemma on $\triangle DFX$ we get $\frac{FR}{RX} = \frac{DF}{DX} \cdot \frac{FC}{CE}$ and by Ratio Lemma on $\triangle AXE$ we get $\frac{AX}{AE} \cdot \frac{FB}{BE}$. Putting this all together, it suffices to show

$$\frac{CE}{CF} \cdot \frac{EB}{BF} \cdot \frac{DF}{DX} \cdot \frac{FC}{CE} \cdot \frac{AX}{AE} \cdot \frac{FB}{BE} = -1.$$

Most of these terms cancel out leaving us with

$$\frac{DF}{DX} \cdot \frac{AX}{AE} \stackrel{?}{=} 1$$

which is true as $\triangle DFX \sim \triangle AEX$. $\square$

5 Problems

A Problems

A1. (Gergonne point) Let $\triangle DEF$ be the contact triangle of $\triangle ABC$. Prove that AD, BE, and CF are concurrent.

A2. Let AD, BE, and CF be cevians in $\triangle ABC$ which all intersect at a point P. Prove that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

A3. Let M, E, and F be points on BC, CA, and AB. Prove that $EF \parallel BC$ if and only if M is the midpoint of BC.

A4. Let $\triangle ABC$ be a triangle and let α be some fixed angle in $[0, \pi)$. Let A_1 be the point such that $\angle BA_1C = \alpha$, $A_1B = A_1C$, and A_1 is on the opposite of BC as A. Define B_1 and C_1 similarly. Prove that AA_1, BB_1, and CC_1 are concurrent.

A5. Let AD, BE, and CF be cevians in $\triangle ABC$ which are concurrent. The circumcircle of $\triangle DEF$ intersects BC, CA, and AB again at D', E' and F' respectively. Prove that AD', BE', and CF' are concurrent.

A6. (CMO 2008) $ABCD$ is a convex quadrilateral for which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

A7. (Euclid 2012) Let $ABCD$ be a convex quadrilateral and let M and N be points on segments AB and CD respectively such that $\frac{AM}{AB} = \frac{CN}{CD}$. Line segments AN and MD intersect at point P and BN and CM intersect at point Q. Prove that $[PMQN] = [BQC] + [APD]$.

A8. (Lemoine axis) Let Γ be the circumcircle of $\triangle ABC$. The tangents to Γ at A, B, and C intersect BC, CA, and AB at points D, E, and F respectively. Prove that D, E, and F are collinear.

A9. (CMOQR 2013) In triangle ABC, $\angle CAB = 90^\circ$ and $\angle BCA = 70^\circ$. Let F be the point on AB such that $\angle ACF = 30^\circ$ and let E be the point on CA such that $\angle CFE = 20^\circ$. Prove that BE bisects $\angle ABC$.

A10. (2016 AIME) Triangle ABC is inscribed in circle ω. Points P and Q are on side $\overline{AB}$ with $AP < AQ$. Rays CP and CQ meet ω again at S and T (other than C), respectively. If $AP = 4, PQ = 3, QB = 6, BT = 5$, and $AS = 7$, find ST.

A11. (Symmedians) Let X be the intersection of the tangents to the circumcircle of $\triangle ABC$ at B and C. Prove that AX is isogonal to the A-median with respect to $\angle CAB$.

A12. (Monge's Theorem) Let Γ_1, Γ_2, and Γ_3 be three circles in the plane such that none of the circles contains another. For each pair, construct the intersection of their common external tangents. Prove that these three intersections are collinear.

A13. (Exeter point) Let $\triangle ABC$ be a triangle. Let the medians through A, B, and C meet the circumcircle of $\triangle ABC$ at A' , B', and C' respectively. Let $\triangle DEF$ be the triangle formed by the tangents at A, B, and C to the circumcircle of $\triangle ABC$. Prove that DA', EB', and FC' are concurrent.

A14. Let $ABCD$ be a cyclic quadrilateral. Prove that

$$AB \cdot BD \cdot DA + BC \cdot CD \cdot DB = AC \cdot CD \cdot DA + AB \cdot BC \cdot CA.$$

B Problems

B1. (CMO 1994) Let ABC be an acute triangle and let AD be the altitude on BC (D on BC). Let H be any point on AD. Lines BH and CH, when extended, intersect AC and AB at E and F, respectively. Prove that $\angle EDH = \angle FDH$.

B2. Let the incircle (I) of a triangle ABC touch BC, CA and AB at D, E and F respectively. Denote by M, N and P the $A-$ excenter, $B-$ excenter and $C-$ excenter of the triangle ABC respectively. Prove that the circles $(IDM), (IEN)$ and (IFP) are coaxial.

B3. Let ABC be right-angled at A, and let D be a point lying on the side AC. Denote by E the reflection of A over line BD, and by F the intersection of CE with the perpendicular through D to the line BC. Prove that AF, DE, and BC are concurrent.

B4. (ISL 2006) Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

B5. (ISL 2000) Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

B6. (USAMO 2012) Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear.

B7. (Newton-Gauss line) Let $ABCD$ be a convex quadrilateral. Let E be the intersection of AB with CD and F be the intersection of AD with BC. Prove that the midpoints of AC, BD, and EF are collinear.

B8. (USATSTST 2017) Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of $\triangle ABC$.

B9. (APMO 2016) Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P. Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

B10. (USATST 2012) In cyclic quadrilateral $ABCD$, diagonals AC and BD intersect at P. Let E and F be the respective feet of the perpendiculars from P to lines AB and CD. Segments BF and CE meet at Q. Prove that lines PQ and EF are perpendicular to each other.

B11. (China 2002) Let E and F be the intersections of opposite sides of a convex quadrilateral $ABCD$. The two diagonals meet at P. Let O be the foot of the perpendicular from P to EF. Show that $\angle BOC = \angle AOD$.

C Problems

C1. (USATST 2021) Let A, B, C, D be four points such that no three are collinear and D is not the orthocenter of ABC. Let P, Q, R be the orthocenters of $\triangle BCD$, $\triangle CAD$, $\triangle ABD$, respectively. Suppose that the lines AP, BQ, CR are pairwise distinct and are concurrent. Show that the four points A, B, C, D lie on a circle.

C2. (USATST 2020) Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

C3. (ISL 2011) Let ABC be a triangle with $AB = AC$ and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B and C at the point E inside the triangle ABC. The line BD intersects the circle through A, E and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incentre of triangle KAB.

C4. (USAMO 2017) In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90^\circ$.

C5. (China 2015) Let A, B, D, E, F, C be six points lie on a circle (in order) such that $AB = AC$. Let $P = AD \cap BE, R = AF \cap CE, Q = BF \cap CD, S = AD \cap BF, T = AF \cap CD$. Let K be a point lie on ST satisfy $\angle QKS = \angle ECA$. Prove that $\frac{SK}{KT} = \frac{PQ}{QR}$.

C6. (USAMO 2016) Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $\overleftrightarrow{I_BF}$ and $\overleftrightarrow{I_CE}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

C7. (USATST 2015) Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively. Let S be a point lying on the Euler line. Denote by X, Y, Z the second intersections of M_aS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.