Functional Equations

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Stuff you probably already know:

- **Injectivity**
  A function f is injective if $f(a) = f(b) \implies a = b$. It's great. If you can manipulate things well, you'll be able to unwrap f. Even partial results can be very helpful (e.g. finding conditions on (a, b) for $f(a) = f(b)$ or proving injectivity for a subset of the range/domain).
- **Surjectivity**
  A function f is surjective if for any y in the range, there is an x for which $f(x) = y$. Not as useful as injectivity, but you should still keep it in mind. Uses of surjectivity are either very easy or very tricky.
- **Induction**
  Not really in this problemset as it's mainly focused on real domain problems. However, you should know that Cauchy's functional equation can still apply if there are additional constraints (e.g. boundedness, monotonicity).

Some mechanical tricks:

- **Involution Trick**
  Given $f(f(x)) = A(x)$, the involution trick is composing again with f. Then you get $f(A(x)) = f(f(f(x))) = A(f(x))$.
- **Forced Cancellation**
  Try plugging in values to set terms on both sides equal or to induce some cancellation.
- **(Lack of) Symmetry**
  If you've plugged in x and y, what about y and x? If there are symmetric terms, you can cancel them out.
- **Transformation**
  Sometimes, you can define a function g in terms of f which has nicer properties. If you can't prove injectivity or surjectivity for f, you might be able to prove it for a transformation of f.

Other advice:

- Try to figure out what the solution set is early on. Plug in $f(x) = ax^2 + bx + c$ and see what comes out (although sometimes this fails).
- Remember to explicitly verify that the solutions work. You may lose a point otherwise.
- Watch out for pathological (i.e. weird and counterintuitive) solutions.
- In a similar vein, don't fall into the pointwise trap. If you've proved that $f(x) = A(x)$ or $f(x) = B(x)$ for every point x, you haven't actually shown that $f \equiv A$ or $f \equiv B$!

A Problems

A1. (IMO 2010). Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

A2. (ELMO 2014). Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x+f(y)) &= g(x)+h(y) \\ g(x+g(y)) &= h(x)+f(y) \\ h(x+h(y)) &= f(x)+g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is injective if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

A3. (USATST 2012). Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for every pair of real numbers x and y,

$$f(x+y^2) = f(x) + |yf(y)|.$$

A4. (USAMO 2016). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x)+xy)\cdot f(x-3y) + (f(y)+xy)\cdot f(3x-y) = (f(x+y))^2.$$

B Problems

B1. (ISL 2018). Determine all functions $f : (0, \infty) \to \mathbb{R}$ satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all $x, y > 0$.

B2. (ISL 2016). Find all functions $f : (0, \infty) \to (0, \infty)$ such that for any $x, y \in (0, \infty)$,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right).$$

B3. (IMO 2013). Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;

(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;

(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

B4. (USAMO 2018). Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

B5. (HMIC 2018). Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + f(y + xy)) = (y+1)f(x+1) - 1$$

for all $x, y \in \mathbb{R}^+$.

B6. (Putnam 2016). Find all functions f from the interval $(1, \infty)$ to $(1, \infty)$ with the following property: if $x, y \in (1, \infty)$ and $x^2 \leq y \leq x^3$, then $(f(x))^2 \leq f(y) \leq (f(x))^3$.

B7. (APMO 2011). Determine all functions $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, satisfying the following two conditions:

1) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.
2) For every pair of real numbers x and y,

$$f(xf(y)) + yf(x) = xf(y) + f(xy)$$

is satisfied.

B8. (IMO 2015). Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

C Problems

C1. (APMO 2019). Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = f(f(x)) + f(y^2) + 2f(xy)$$

for all real numbers x and y.

C2. (ISL 2004). Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x^2 + y^2 + 2f(xy)) = (f(x + y))^2$$

for all $x, y \in \mathbb{R}$.

C3. (IMO 2011). Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies

$$f(x + y) \leq yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

C4. (IMO 2017). Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(f(x)f(y)) + f(x + y) = f(xy).$$

C5. (ELMO 2017). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers a, b, and c:

(i) If $a + b + c \geq 0$ then $f(a^3) + f(b^3) + f(c^3) \geq 3f(abc)$.

(ii) If $a + b + c \leq 0$ then $f(a^3) + f(b^3) + f(c^3) \leq 3f(abc)$.

C6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$ and

$$f\left(f(x)y + \frac{x}{y}\right) = xyf\left(x^2 + y^2\right)$$

for all real numbers x and y with $y \neq 0$.