Functional Equations Over Reals

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1 Introduction

Functional equations (FEs) are a very flexible family of problems. The basic setup is the same:

Find all $f : A \to B$ such that
$$\cdots = \cdots$$
for all $x, y, \ldots \in A$.

However, two problems that are almost the exact same can have wildly different approaches needed to solve them. We will discuss common themes, but the best way to develop a sense for FEs is to solve a good variety of them. Let's take a look at a problem.

Example 1 (Iran 1996)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

When approaching an FE, here are a few things to try to get a general feel for the problem:

- See if you can guess the solution set early on! You can do this by checking common functions such as $f(x) \equiv c$, $f(x) \equiv ax + b$, $f(x) \equiv cx^2$, etc. This will also help inform what properties it might satisfy, and that you could potentially prove.
- If you have the solution set, that's likely worth a point! Make sure to show that it works though, or you could get docked.
- Try to identify helpful values such as $f(0)$.
- Make things cancel! You have a lot of leeway with choosing what to plug-in, and some choices can greatly simplify the assertion.
- Related to the above, look for symmetric terms to cancel.
- If applicable, show that the function is odd/even.
- Some difficult problems will require analyzing pre-images such as $\{a \mid f(a) = 0\}$.

Proof. Let $P(x, y)$ denote the assertion. For this example, by trying the common solutions, we see that two of them happen to work: $f(x) \equiv 0$ and $f(x) \equiv x^2$. Let's explicitly verify these two. If $f(x) \equiv 0$, then

$$\begin{aligned} P(x,y) \implies \quad & 0 = 0 + 0 \cdot y \\ \iff \quad & 0 = 0. \end{aligned}$$

If $f(x) \equiv x^2$, then

$$\begin{aligned} P(x,y) \implies & \quad (x^2+y)^2 = (x^2-y)^2 + 4x^2y \\ \iff & \quad x^4 + 2x^2y + y^2 = x^4 - 2x^2y + y^2 + 4x^2y. \end{aligned}$$

So both work.

Now let's try to prove that these are the only two solutions. Plugging a few things in, we get

$$\begin{aligned} P(0,0) \implies & \quad f(f(0)) = f(0) \\ P(0,-f(0)) \implies & \quad f(0) = f(f(0)) - 4f(0)^2 \\ \implies & \quad 4f(0)^2 = 0 \\ \implies & \quad f(0) = 0 \end{aligned}$$

This is useful! Now that we know $f(0) = 0$, let's try to cancel terms by setting $f(x) + y = 0$ and $x^2 - y = 0$.

$$\begin{aligned} P(x,-f(x)) \implies & \quad 0 = f(x^2 + f(x)) - 4f(x)^2 \\ P(x,x^2) \implies & \quad f(f(x) + x^2) = 4x^2 f(x) \\ \implies & \quad 4f(x)^2 = 4x^2 f(x). \end{aligned}$$

Thus, we have that $f(x) = 0$ or $f(x) = x^2$. At first glance, it may seem that we are done. However, this only proves that $f(x) \in \{0, x^2\}$ for each individual x. What we want to show is that $f(x) = 0 \,\forall\, x \in \mathbb{R}$, or $f(x) = x^2 \,\forall\, x \in \mathbb{R}$. This is known as the **pointwise trap**. Luckily, it's not too difficult to resolve it in this problem. We will do casework on $f(1)$.

If $f(1) = 1$, then $P(1,y)$ gives $f(1+y) = f(1-y) + 4y$. It is easy to check that if $y \neq 0$, we must have $f(1+y) = (1+y)^2$ and $f(1-y) = (1-y)^2$. So $f(x) = x^2$ for all $x \in \mathbb{R}$.

If $f(1) = 0$, then $P(1,y)$ gives $f(y) = f(1-y)$. This gives us that $f(y) = 0$ unless $y^2 = (1-y)^2$. This tells us that $f(2) = 0$, so $P(2,y)$ gives $f(y) = f(4-y)$. Thus, $f(y) = 0$ unless $y^2 = (4-y)^2$. Together, we must have $f(y) = 0$ for all $y \in \mathbb{R}$. □

Now that we've gotten a taste for FEs, let's take a look at some useful ideas for tackling some tougher problems. This handout specifically focuses on FEs over $\mathbb{R}$ with a given equation condition. I would also recommend checking out Evan Chen's handout and James Rickards's handout, both of which I drew inspiration and examples from.

2 Injectivity

Definition (*jectivity)**.** A function $f : A \to B$ is

- **injective** if $f(x) = f(y) \iff x = y$
- **surjective** if for any $b \in B$, $\exists\, a \in A$ such that $f(a) = b$
- **bijective** if f is both injective and surjective

Injectivity won't be applicable to every problem, but when it is, it is a very powerful tool. With proper manipulation, injectivity can allow us to unwrap f, as seen in the following example.

Example 2 (ISL 2002)**.** Find all functions f from the reals to the reals such that

$$f\left(f(x)+y\right) = 2x + f\left(f(y)-x\right)$$

for all real x, y.

Often, we don't even need injectivity for all values. Pointwise injectivity alone can be very useful.

Example 3 (Japan 2024)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + xy) = f(x)f(x + y)$$

for all real numbers x, y.

To leverage injectivity, we need an equation with a single f term on both sides. Sometimes, we need to be clever in order to transform our given assertion into that form. The following is a difficult problem that feels like a breeze once the right things are plugged in.

Example 4 (HMIC 2018)**.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + f(y + xy)) = (y + 1)f(x + 1) - 1$$

for all $x, y \in \mathbb{R}^+$.

2.1 Problems

1. (Ukraine 2017). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x + f(f(y))) = y + f(f(x))$$
for all real numbers x and y.
2. (EGMO 2012). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f\left(yf(x + y) + f(x)\right) = 4x + 2yf(x + y)$$
for all $x, y \in \mathbb{R}$.
3. (ELMO 2014). Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying
$$\begin{aligned} f(x + f(y)) &= g(x) + h(y) \\ g(x + g(y)) &= h(x) + f(y) \\ h(x + h(y)) &= f(x) + g(y) \end{aligned}$$
for all real numbers x and y.
4. (APMO 2019). Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 + f(y)) = f(f(x)) + f(y^2) + 2f(xy)$$
for all real numbers x and y.
5. (IMO 2017). Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,
$$f(f(x)f(y)) + f(x + y) = f(xy).$$

3 Cauchy's Functional Equation

The Cauchy FE is an important functional equation to be aware of, as many problems can be reduced to it. Note that its behaviour is different when it is over $\mathbb{R}$ versus when it is over $\mathbb{Z}$ or $\mathbb{Q}$. Over $\mathbb{R}$, the Cauchy FE is not enough to guarantee that the function is linear, whereas it is enough for $\mathbb{Z}$ and $\mathbb{Q}$.

Theorem 1 (Cauchy's Functional Equation)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is **additive** if $f(x + y) = f(x) + f(y)$ for any $x, y \in \mathbb{R}$. For such an f, $f(qx) = qf(x)$ for any $q \in \mathbb{Q}$. Additionally, if f is additive and at least one of {continuous, bounded, monotonic}, then f is linear.

Proof. We will not prove the full statement. To demonstrate the general idea of the proof, let's consider $f : \mathbb{R}^+ \to \mathbb{R}^+$ and f additive. This is an example of the bounded case. We will assume that $f(qx) = qf(x)$ for any $q \in \mathbb{Q}^+$ is already proved for us. In particular, we have $f(q) = cq$ for some constant c and any $q \in \mathbb{Q}^+$.

Our goal is to show that $f(x) = cx$ for all $x \in \mathbb{R}^+$. Assume for the sake of contradiction that there is some t such that $f(t) \neq ct$. Say $f(t) > ct$ and let $L = f(t) - ct$ (the other case is similar). For any $\epsilon > 0$, we can find a $q \in \mathbb{Q}^+$ such that $|q - t| < \epsilon$, as the rationals are dense. Let us choose ϵ so that $\frac{\epsilon}{t} \ll \frac{L}{f(t)}$. The intuition is that the slope from $(t, f(t))$ to $(q, f(q))$ is very large in absolute value, and following this line will lead to a negative value.

Indeed, take $k > \left\lceil \frac{f(t)}{L} \right\rceil$, $k \in \mathbb{N}$. Then define $s := t + (q - t)k$. We can check that $s > 0$ as $k\epsilon < t$ so s is a valid input to f. Then

$$\begin{aligned} f(s) &= f(t) + kf(q - t) \\ &= f(t) + k(f(q) - f(t)) \\ &= f(t) + k(cq - ct - L) \\ &\leq f(t) + k\epsilon - kL \\ &< 0. \end{aligned}$$

As f takes on positive values, we have a contradiction. Thus, $f(x) = cx$ for all $x \in \mathbb{R}^+$. □

Let's take a look at an example problem which makes use of the Cauchy FE.

Example 5 (Belarus 2014)**.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(f(x) + y) = x + f(y),$$

for all $x, y \in \mathbb{R}^+$.

Often, you don't need these specific properties to go from additive to linear. With a bit of work, other conditions can help bridge the gap.

Example 6 (Field Automorphisms of $\mathbb{R}$)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\forall\, x, y \in \mathbb{R}$, we have

$$f(x + y) = f(x) + f(y) \text{ and } f(xy) = f(x)f(y).$$

3.1 Transformations

Sometimes, you can define a function g in terms of f which has nicer properties. If you can't prove injectivity or surjectivity for f, you might be able to prove it for a transformation of f. It may also be useful if it simplifies the given equation.

Example 7 (Jensen's Functional Equation)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ which are one of $\{$continuous, bounded, monotonic$\}$ and which satisfy the equation

$$f(x) + f(y) = 2f\left(\frac{x+y}{2}\right)$$

for any $x, y \in \mathbb{R}$.

Example 8 (ISL 2007)**.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f\left(x + f\left(y\right)\right) = f\left(x + y\right) + f\left(y\right)$$

for all pairs of positive reals x and y. Here, $\mathbb{R}^+$ denotes the set of all positive reals.

3.2 Problems

1. (CMO 2008). Determine all functions f defined on the set of rational numbers that take rational values for which
$$f(2f(x) + f(y)) = 2x + y,$$
for each x and y.
2. (USATST 2012). Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for every pair of real numbers x and y,
$$f(x + y^2) = f(x) + |yf(y)|.$$
3. (USATSTST 2019). Find all binary operations $\diamondsuit : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning $\diamondsuit$ takes pairs of positive real numbers to positive real numbers) such that for any real numbers $a, b, c > 0$,
   - the equation $a \diamondsuit (b \diamondsuit c) = (a \diamondsuit b) \cdot c$ holds;
   - and if $a \geq 1$ then $a \diamondsuit a \geq 1$.
4. (USAMO 2018). Find all functions $f : (0, \infty) \to (0, \infty)$ such that
$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$
for all $x, y, z > 0$ with $xyz = 1$.
5. (ELMOSL 2013). Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $f(x) + f(y) = f(x + y)$ and $f(x^{2013}) = f(x)^{2013}$.

4 Functional Inequalities

The ideas for functional inequalities are often similar, but tend to be more ad hoc and problem-dependent. Often times, the inequalities are pushed to equality.

Example 9 (IMO 2022)**.** Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leq 2.$$

4.1 Positive Reals

Often times, the domain and range of the functional equation create implicit inequalities. Functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ are particularly susceptible to such ideas.

Example 10 (Iran 2019)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:

$$f\left(f(x)^2 - y^2\right)^2 + f(2xy)^2 = f\left(x^2 + y^2\right)^2$$

4.2 Problems

1. (Netherlands 2023). Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which
$$f(a-b)f(c-d) + f(a-d)f(b-c) \leq (a-c)f(b-d),$$
for all real numbers a, b, c and d. Note that there is only one occurrence of f on the right hand side!

2. (USAMO 2000). Call a real-valued function f very convex if
$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x - y|$$
holds for all real numbers x and y. Prove that no very convex function exists.

3. (APMO 2023). Let $c > 0$ be a given positive real and $\mathbb{R}_{>0}$ be the set of all positive reals. Find all functions $f \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that
$$f((c+1)x + f(y)) = f(x + 2y) + 2cx \quad \text{for all } x, y \in \mathbb{R}_{>0}.$$

4. (IMO 2013). Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:
   (i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;
   (ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;
   (iii) there exists a rational number $a > 1$ such that $f(a) = a$.

   Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

5. (IMO 2011). Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on the set of real numbers that satisfies
$$f(x+y) \leq yf(x) + f(f(x))$$
for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

6. (ISL 2020). Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive real numbers x and y :

$$f(x + f(xy)) + y = f(x)f(y) + 1.$$

5 Problems

A1. (Putnam 2008). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for real numbers $x, y,$ and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

A2. (CMOR 2016). Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) + f(x - f(y)) = x.$$

A3. (Indonesia 2022). Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$ we have

$$f(f(f(x)) + f(y)) = f(y) - f(x).$$

A4. (IMO 2010). Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

A5. (USAMO 2002). Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

B1. (ISL 2016). Find all functions $f : (0, \infty) \to (0, \infty)$ such that for any $x, y \in (0, \infty)$,

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right).$$

B2. (USAMO 2016). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

B3. (APMO 2002). Let $\mathbb{R}$ denote the set of all real numbers. Find all functions f from $\mathbb{R}$ to $\mathbb{R}$ satisfying:

(i) there are only finitely many s in $\mathbb{R}$ such that $f(s) = 0$, and

(ii) $f(x^4 + y) = x^3 f(x) + f(f(y))$ for all x, y in $\mathbb{R}$.

B4. (ISL 2022). Let $\mathbb{R}$ be the set of real numbers. We denote by $\mathcal{F}$ the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = f(x) + f(y)$$

for every $x, y \in \mathbb{R}$ Find all rational numbers q such that for every function $f \in \mathcal{F}$, there exists some $z \in \mathbb{R}$ satisfying $f(z) = qz$.

B5. (ISL 2018). Determine all functions $f : (0, \infty) \to \mathbb{R}$ satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all $x, y > 0$.

B6. (APMO 2011). Determine all functions $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers, satisfying the following two conditions:

(i) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.

(ii) For every pair of real numbers x and y,

$$f(xf(y)) + yf(x) = xf(y) + f(xy)$$

is satisfied.

C1. (IMO 2015). Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

C2. (ISL 2004). Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x^2 + y^2 + 2f(xy)) = (f(x + y))^2$$

for all $x, y \in \mathbb{R}$.

C3. (USATST 2024). Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$f(xf(y)) + f(y) = f(x + y) + f(xy).$$

C4. (ELMO 2017). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers a, b, and c:

(i) If $a + b + c \geq 0$ then $f(a^3) + f(b^3) + f(c^3) \geq 3f(abc)$.

(ii) If $a + b + c \leq 0$ then $f(a^3) + f(b^3) + f(c^3) \leq 3f(abc)$.

C5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$ and

$$f\left(f(x)y + \frac{x}{y}\right) = xyf\left(x^2 + y^2\right)$$

for all real numbers x and y with $y \neq 0$.

6 Exercise Solutions

6.1 Introduction Solutions

Example 1 (Iran 1996)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

Proof. See main handout. □

6.2 Injectivity Solutions

Example 2 (ISL 2002)**.** Find all functions f from the reals to the reals such that

$$f\left(f(x) + y\right) = 2x + f\left(f(y) - x\right)$$

for all real x, y.

Proof. Let $P(x, y)$ denote the assertion. We will first show f is surjective.

$$P(x, -f(x)) \implies \qquad f(f(-f(x)) - x) = f(0) - 2x$$

As $f(0) - 2x$ can take on any real value, we see that f is surjective. Now we will show that f is injective. Assume that $f(a) = f(b)$ for some a, b. By surjectivity, we can choose k such that $f(k) = a + b$. Then

$$\begin{aligned} P(a, k) &\implies \qquad f(f(a) + k) = 2a + f(b) \\ P(b, k) &\implies \qquad f(f(b) + k) = 2b + f(a). \end{aligned}$$

So $a = b$ and hence f is injective. Now I claim that injectivity is enough to finish.

$$\begin{aligned} P(0, y) &\implies \qquad f(f(0) + y) = f(f(y)) \\ &\implies \qquad f(y) = y + f(0). \end{aligned}$$

We can indeed verify that $f(y) \equiv y + c$ is a solution for any constant $c \in \mathbb{R}$.

$$\begin{aligned} P(x, y) &\implies \qquad f(x + c + y) = 2x + f(y + c - x) \\ &\iff \qquad x + y + 2c = 2x + y - x + 2c. \end{aligned}$$

□

Example 3 (Japan 2024)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + xy) = f(x)f(x + y)$$

for all real numbers x, y.

Proof. Let $P(x, y)$ denote the assertion.

$$P(0, x) \implies \quad f(f(0)) = f(0)f(x).$$

If f is constant, then we can easily see that $f(x) \equiv 0$ or $f(x) \equiv 1$. Otherwise, we must have $f(0) = 0$. Now assume for the sake of contradiction that $f(a) = 0$ for some $a \neq 0$. Then

$$P(a, x) \implies \quad f(ax) = 0.$$

This implies a constant solution, so contradiction. So 0 is the only zero of f. Now consider

$$\begin{aligned} P(x, -x) \implies& \quad f(f(x) - x^2) = 0 \\ \implies& \quad f(x) - x^2 = 0. \end{aligned}$$

So we find that the only non-constant solution is $f(x) = x^2$. We can verify this easily:

$$\begin{aligned} P(x, y) \implies& \quad f(x^2 + xy) = x^2(x + y)^2 \\ \iff& \quad (x^2 + xy)^2 = x^2(x + y)^2. \end{aligned}$$

□

Example 4 (HMIC 2018)**.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + f(y + xy)) = (y + 1)f(x + 1) - 1$$

for all $x, y \in \mathbb{R}^+$.

Proof. Let $P(x, y)$ be the assertion. We will first prove that f is injective. Assume that $f(a) = f(b)$ for some a, b. Then

$$\begin{aligned} P\left(1, \frac{a}{2}\right) \implies& \quad f(1 + f(a)) = (a + 1)f(2) - 1 \\ P\left(1, \frac{b}{2}\right) \implies& \quad f(1 + f(b)) = (b + 1)f(2) - 1 \\ \implies& \quad (a + 1)f(2) - 1 = (b + 1)f(2) - 1 \\ \implies& \quad a = b. \end{aligned}$$

Now our goal is to make use of the injectivity. Note that the RHS of the assertion can be expanded as

$$f(x + 1) + yf(x + 1) - 1$$

while the LHS is a single f term. If we could isolate the $f(x+1)$ in the RHS, we could unwrap the f's on both sides. This motivates the next step.

$$\begin{aligned}
P\left(x, \frac{1}{f(x+1)}\right) \implies &\quad f\left(x+f\left(\frac{x+1}{f(x+1)}\right)\right) = f(x+1) + \frac{f(x+1)}{f(x+1)} - 1 \\
\implies &\quad f\left(x+f\left(\frac{x+1}{f(x+1)}\right)\right) = f(x+1) \\
\implies &\quad x+f\left(\frac{x+1}{f(x+1)}\right) = x+1 \\
\implies &\quad f\left(\frac{x+1}{f(x+1)}\right) = 1.
\end{aligned}$$

As the last line is true for all $x \in \mathbb{R}^+$, we in fact have

$$f\left(\frac{x+1}{f(x+1)}\right) = f\left(\frac{y+1}{f(y+1)}\right)$$

for all $x, y \in \mathbb{R}^+$. Appealing to injectivity once again, we see that for any $x, y \in \mathbb{R}^+$,

$$\frac{x+1}{f(x+1)} = \frac{y+1}{f(y+1)}.$$

Equivalently, $f(x) = cx$ for some constant c and $x > 1$. From $P(x, y)$ with $x, y > 1$, we see that $c = 1$.

It remains to show that $f(x) = x$ for $x \in (0, 1]$. From $f\left(\frac{x+1}{f(x+1)}\right) = 1$, we see that $f(1) = 1$. For $x \in (0, 1)$, consider

$$\begin{aligned}
P\left(1-x, \frac{x}{2-x}\right) \implies &\quad f\left(1-x+f\left(\frac{((1-x)+1)x}{2-x}\right)\right) = \frac{2}{2-x} \cdot f(2-x) - 1 \\
\implies &\quad f(1-x+f(x)) = \frac{2}{2-x} \cdot (2-x) - 1 \\
\implies &\quad f(1-x+f(x)) = 1 \\
\implies &\quad f(1-x+f(x)) = f(1) \\
\implies &\quad 1-x+f(x) = 1 \\
\implies &\quad f(x) = x.
\end{aligned}$$

We can check that $f(x) \equiv x$ indeed works. □

6.3 Cauchy's FE Solutions

Example 5 (Belarus 2014)**.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(f(x) + y) = x + f(y),$$

for all $x, y \in \mathbb{R}^+$.

Proof. Let $P(x, y)$ denote the assertion. We interpret $f(f(y + f(x)) + y)$ in two different ways. Firstly,

$$P(y + f(x), y) \implies \qquad f(f(y + f(x)) + y) = y + f(x) + f(y)$$

From $P(x, y)$, we also have $f(y + f(x)) = x + f(y)$. So

$$\begin{aligned} & f(f(y + f(x)) + y) = f(x + y + f(y)) \\ P(y, x + y) \implies \qquad & f(x + y + f(y)) = y + f(x + y). \end{aligned}$$

This gives us

$$y + f(x) + f(y) = f(f(y + f(x)) + y) = f(x + y + f(y)) = y + f(x + y).$$

Hence, $f(x + y) = f(x) + f(y)$ and f is additive. Since f is also bounded, by theorem 1, $f(x) \equiv cx$ for some $c > 0$. We can check to see that $c = 1$ is the only solution. □

Example 6 (Field Automorphisms of $\mathbb{R}$)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\forall\, x, y \in \mathbb{R}$, we have

$$f(x + y) = f(x) + f(y) \text{ and } f(xy) = f(x)f(y).$$

Proof. The first equation gives additivity of f. We will show that we also have monotonicity. Let P denote the first assertion and Q denote the second. Then $Q(x, x)$ gives $f(x^2) = f(x)^2$. For $y > x$, $P(x, y - x)$ gives

$$f(y) = f(x) + f(y - x) = f(x) + f(\sqrt{y - x})^2 \geq f(x).$$

Thus, we have monotonicity and by Theorem 1, $f(x) \equiv cx$ for some constant c. We can check that $c = 0, 1$ are the only solutions. □

Example 7 (Jensen's Functional Equation)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ which are one of {continuous, bounded, monotonic} and which satisfy the equation

$$f(x) + f(y) = 2f\left(\frac{x + y}{2}\right)$$

for any $x, y \in \mathbb{R}$.

Proof. Define $g(x) := f(x) - f(0)$. This gives us $g(0) = 0$ and

$$g(x) + g(y) = 2g\left(\frac{x + y}{2}\right).$$

In particular, plugging in $y = 0$ gives $g(x) = 2g\left(\frac{x}{2}\right)$. So $g(x) + g(y) = g(x + y)$ and so g is additive. Note also that g is also continuous/bounded/monotonic. Thus, $g(x) = ax$ and hence $f(x) = ax + b$ and it can be checked that any $a, b \in \mathbb{R}$ work. □

Example 8 (ISL 2007)**.** Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f\left(x + f\left(y\right)\right) = f\left(x + y\right) + f\left(y\right)$$

for all pairs of positive reals x and y. Here, $\mathbb{R}^+$ denotes the set of all positive reals.

Proof. Let $P(x, y)$ be the assertion. Define $g(x) := f(x) - x$. Then we can rewrite the assertion as

$$\begin{gathered} g(x + g(y) + y) + x + g(y) + y = g(x + y) + x + y + g(y) + y \\ g(x + y + g(y)) = g(x + y) + y. \end{gathered}$$

We will show that $g(x) \geq 0$ for all $x \in \mathbb{R}^+$. Assume for the sake of contradiction that $g(a) < 0$ for some $a \in \mathbb{R}^+$. Then

$$\begin{aligned} P(-g(a), a) \implies & \quad g(-g(a) + a + g(a)) = g(-g(a) + a) + a \\ \implies & \quad g(-g(a) + a) + a - g(a) = 0 \\ \implies & \quad f(a - g(a)) = 0. \end{aligned}$$

This is a contradiction, as f takes on positive values. Now, let us prove that g is injective. Assume $g(a) = g(b)$ for some a, b.

$$\begin{aligned} P(a, b) \implies & \quad g(a + b + g(b)) = g(a + b) + b \\ P(b, a) \implies & \quad g(a + b + g(a)) = g(a + b) + a \\ \implies & \quad a = b. \end{aligned}$$

So g is injective. Now we will show that g is additive.

$$\begin{aligned} P(x, s + t) \implies & \quad g(x + s + t + g(s + t)) = g(x + s + t) + s + t \\ P(x + t, s) \implies & \quad g(x + s + t + g(s)) = g(x + s + t) + s \\ \implies & \quad g(x + s + t + g(s)) + t = g(x + s + t) + s + t \\ P(x + s + g(s), t) \implies & \quad g(x + s + t + g(s) + g(t)) = g(x + s + t + g(s)) + t. \end{aligned}$$

Combining these results gives

$$\begin{gathered} g(x + s + t + g(s + t)) = g(x + s + t + g(s) + g(t)) \\ x + s + t + g(s + t) = x + s + t + g(s) + g(t) \\ g(s + t) = g(s) + g(t). \end{gathered}$$

As g is additive and bounded, Theorem 1 tells us that $g(x) \equiv cx$ for some $c \geq 0$. We can check that $c = 1$, which corresponds to the solution $f(x) \equiv 2x$. ☐

6.4 Inequality Solutions

Example 9 (IMO 2022)**.** Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leq 2.$$

Proof. Let $P(x, y)$ be the assertion, with the direction of the inequality given by the context. For any $x \in \mathbb{R}^+$, let $t(x)$ to be the unique $y \in \mathbb{R}^+$ such that the inequality holds. Clearly, $t(t(x)) = x$. We will first show that $t(x) = x$ for all x. Assume otherwise, that there is some $a \neq b$ such that $t(a) = b$. Then since $t(a) = b \neq a$ and $t(b) = a \neq b$,

$$\begin{aligned}
P(a,a) &\implies & 2af(a) &> 2 \\
&\implies & f(a) &> \frac{1}{a} \\
P(b,b) &\implies & 2bf(b) &> 2 \\
&\implies & f(b) &> \frac{1}{b} \\
P(a,b) &\implies & 2 &\geq af(b) + bf(a) \\
&& &> \frac{a}{b} + \frac{b}{a} \\
&& &\geq 2.
\end{aligned}$$

So there is a contradiction and we must have $t(x) = x$. So

$$P(x,x) \implies \quad f(x) \leq \frac{1}{x}.$$

Let us now show that $f(x) = \frac{1}{x}$. Assume for the sake of contradiction that there is some a such that $f(a) = \frac{c}{a}$ for $c < 1$. Then consider $x = a, y = a + \epsilon \neq t(a)$ for some $\epsilon > 0$. We have

$$P(a, a+\epsilon) \implies 2 < af(a+\epsilon) + (a+\epsilon)f(a) \leq \frac{a}{a+\epsilon} + \frac{c(a+\epsilon)}{a}.$$

For sufficiently small ϵ, the RHS becomes less than 2. This can be seen as the RHS is continuous as a function of ϵ, and when $\epsilon = 0$, the RHS is $1 + c < 2$. So we must have $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}^+$.

It remains to verify that this solution satisfies the problem. Indeed, for $x = y$, we have

$$2xf(x) = 2 \leq 2.$$

For $x \neq y$, we have

$$xf(y) + yf(x) = \frac{x}{y} + \frac{y}{x} = 2 + \left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^2 > 2.$$

So $f(x) = \frac{1}{x}$ works. □

Example 10 (Iran 2019)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:

$$f\left(f(x)^2 - y^2\right)^2 + f(2xy)^2 = f\left(x^2 + y^2\right)^2$$

Proof. Let $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be defined as $g(x) = f(x)^2$. Let $P(x, y)$ be our assertion, which can now be written as

$$P(x,y) \implies \quad g(g(x) - y^2) + g(2xy) = g(x^2 + y^2).$$

From $P(x, y)$ and $P(x, -y)$, we see that $g(2xy) = g(-2xy)$. So g is even. Trying $P(x, x)$ gives

$$g(g(x) - x^2) = 0.$$

If we had injectivity at 0, we would be done. This motivates us to consider the set of zeros $Z := \{x \in \mathbb{R} \mid g(x) = 0\}$. The key claim will be that $z \in Z$ and $|t| \leq |z| \implies t \in Z$. Indeed, let us consider such a pair t, z. Then thanks to the condition $|t| \leq |z|$, there exist $x_0, y_0 \in \mathbb{R}$ which solve the system

$$2x_0y_0 = t, \; x_0^2 + y_0^2 = z.$$

Now

$$\begin{aligned} P(x_0, y_0) \implies & \quad g(g(x_0) - y_0^2) + g(t) = g(z) \\ \implies & \quad g(g(x_0) - y_0^2) + g(t) = 0 \\ \implies & \quad g(t) = 0. \end{aligned}$$

So the claim is proved.

Let $M := \sup Z$. Our claim implies that $Z = (-M, M)$ or $[-M, M]$. If $M = \infty$, then $Z = \mathbb{R} \implies g(x) \equiv 0$ which is a valid solution. Otherwise, M finite. Assume for the sake of contradiction that $M > 0$. Our goal will be to pick x, y so that $|f(x)^2 - y^2| < M$ and $|2xy| < M$, but $x^2 + y^2 > M$. Let $\epsilon > 0$ be such that $\epsilon << M$ and $2\epsilon\sqrt{M} < M$. Let $\delta > 0$ be such that $\epsilon^2 + (\sqrt{M} - \delta)^2 > M$. Then

$$\begin{aligned} P(\epsilon, \sqrt{M} - \delta) \implies & \quad g\left(g(\epsilon) - \left(\sqrt{M} - \delta\right)^2\right) + g\left(2\epsilon\left(\sqrt{M} - \delta\right)\right) = g\left(\epsilon^2 + \left(\sqrt{M} - \delta\right)^2\right) \\ \implies & \quad g\left(-\left(\sqrt{M} - \delta\right)^2\right) = g\left(\epsilon^2 + \left(\sqrt{M} - \delta\right)^2\right) \\ \implies & \quad g\left(\epsilon^2 + \left(\sqrt{M} - \delta\right)^2\right) = 0. \end{aligned}$$

So

$$\epsilon^2 + \left(\sqrt{M} - \delta\right)^2 > M \text{ and } \epsilon^2 + \left(\sqrt{M} - \delta\right)^2 \in Z,$$

contradiction. Thus, $Z = \{0\}$. Now returning to $P(x, x)$, we have

$$g(g(x) - x^2) = 0 \implies g(x) = x^2.$$

This corresponds to $f(x) \in \{x, -x\}$. It is easy to check that this is indeed a valid solution. □