

Real Polynomials

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Polynomials are one of the most ubiquitous mathematical objects, and as such, appear frequently in math olympiads. In this handout, we focus on polynomials with real coefficients and techniques to handle them. This is in contrast to integer polynomials which come with additional number-theoretic structures and ideas, though many of the fundamentals are the same. For practice with integer polynomials, I highly recommend Yufei Zhao's Integer Polynomials handout.

1 The Essentials

We generally analyze polynomials in two forms:

- As a linear combination of powers x^i
- By the roots r_i

The Fundamental Theorem of Algebra tells us that a polynomial has roots and factors into them, while Vieta's Formulas give relations between the roots and the coefficients. It is also important to always consider the degree of the polynomials. If we are able to reduce the degree, for example, we can obtain contradictions based on the number of roots.

Theorem (Fundamental Theorem of Algebra). Let $P(x)$ be a complex polynomial of degree $n \geq 1$. Then $P(x)$ has exactly n complex roots r_1, r_2, \dots, r_n , counting with multiplicity. Furthermore, $P(x) = c \prod_{i=1}^n (x - r_i)$ for some constant c .

Lemma 1.0.1. Let $P(x)$ be a polynomial with real coefficients. Then its non-real roots form pairs of complex conjugates.

Theorem (Vieta's Formulas). Let polynomial $P(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_0$ have roots r_1, r_2, \dots, r_n . Then we have

$$\begin{aligned} r_1 + r_2 + \dots + r_n &= -\frac{c_{n-1}}{c_n}, \\ r_1 r_2 + r_1 r_3 + \dots + r_{n-1} r_n &= \frac{c_{n-2}}{c_n}, \\ &\vdots \\ r_1 r_2 \dots r_n &= (-1)^n \frac{c_0}{c_n}. \end{aligned}$$

Proof. We can obtain these equations by expanding

$$P(x) = c_n(x - r_1)(x - r_2) \dots (x - r_n)$$

and comparing coefficients. □

Additionally, the real polynomials form a Euclidean domain, meaning that the greatest common divisor can be defined and satisfies certain properties.

Definition. For two real polynomials $P(x), Q(x)$, the greatest common divisor $D(x) := \gcd(P, Q)$ is defined as a real polynomial of largest degree which divides both $P(x)$ and $Q(x)$. Additionally, $\deg D \leq \min(\deg P, \deg Q)$.

Remark. Note that the D can be explicitly written based on the shared roots of P, Q , and is unique up to a scalar multiplication.

Theorem (Bezout's Identity). Let $P(x)$ be a real polynomial of degree n and $Q(x)$ be a real polynomial of degree m . Then there exist real polynomials $A(x), B(x)$ such that $A(x)P(x) + B(x)Q(x) = D(x)$, where $D(x) = \gcd(P, Q)$.

Lemma 1.0.2. Let $P(x)$ and $Q(x)$ be real polynomials which are relatively prime. We have

$$P(x) \mid Q(x)R(x) \implies P(x) \mid R(x).$$

Proof. We have $PS = QR$ for some polynomial S . By Bezout's Identity, we also have $AP + BQ = 1$. Then

$$\begin{aligned} PS &= QR \\ BPS &= BQR \\ BPS &= (1 - AP)R \\ (BS + AR)P &= R. \end{aligned}$$

So $P \mid R$. □

2 Handling Conditions

Many problems ask for polynomials satisfying given conditions. In this case, it's always a good idea to consider the degree as it constrains how many conditions the polynomial can reasonably satisfy. The next lemma makes this rigorous.

2.1 Intersecting Polynomials

Lemma 2.1.1. Let $P(x)$ and $Q(x)$ be polynomials which intersect at points x_1, x_2, \dots, x_k . That is, $P(x_i) = Q(x_i)$ for $i = 1, 2, \dots, k$. Then

$$P(x) - Q(x) = A(x) \prod_{i=1}^k (x - x_i) = A(x)(x - x_1)(x - x_2) \dots (x - x_k)$$

for some polynomial $A(x)$ with $\deg A = \deg(P - Q) - k$. If $k > \deg P$ and $k > \deg Q$, $P \equiv Q$.

This fact is quite useful. One particular application is moving from results about integer or discrete inputs to all numbers. For example, if we determine that $P(n) = n$ for all integers n , then we can conclude that $P(x) = x$ for all real numbers x .

Example 1. A polynomial $P(x)$ of degree 10 satisfies $P(k) = \frac{k}{k+1}$ for $k = 0, 1, \dots, 10$. Find $P(11)$.

Proof. If the right hand side were some polynomial expression in terms of k instead, then the problem would be trivial. With this in mind, we transform the problem to force the equation to be a polynomial. Let $Q(x) = (x+1)P(x) - x$. This is clearly a polynomial of degree 11. Then the given equations now become

$$Q(k) = 0$$

for $k = 0, 1, \dots, 10$. So $Q(x) = ax(x-1)(x-2)\dots(x-10)$ for some constant a . To determine this constant, we need one more point on Q . We can find this point by plugging $x = -1$ into $Q(x) = (x+1)P(x) - x$ since this zeros out the unknown $P(x)$ term. Thus,

$$Q(-1) = 1 = a(-1)(-2)(-3)\dots(-11).$$

$$a = \frac{-1}{11!}.$$

Finally, we can compute $Q(11)$. We obtain $Q(11) = -1$, and so $P(11) = \frac{5}{6}$. □

2.2 Finite Differences

If the inputs happen to be at fixed intervals, we can use the method of finite differences.

Lemma 2.2.1. For any polynomial $P(x)$ of degree $k \geq 1$ and constant a , $P(x+a) - P(x)$ is a polynomial of degree $k-1$.

Directly applying Lemma 2.2.1 to P taken at fixed points allows us to recursively form a table of differences.

Theorem (Method of Finite Differences). Let $P(x) = c^n x^n + \dots + c_0$. Say we know the values of $P(1), P(2), \dots, P(m)$ for $m \geq n+1$. Define the difference table so that $D_{r,j} = P(j)$ for $r=0$ and $D_{r,j} = D_{r-1,j+1} - D_{r-1,j}$ for $r \geq 1$. Then row r 's values correspond to a polynomial of degree $n-r$. In particular, $D_{n,j} = n!c_n$ and $D_{n+1,j} = 0$ for all j .

Example 2 (HMMT 2015). The complex numbers x, y, z satisfy

$$\begin{aligned}xyz &= -4 \\(x+1)(y+1)(z+1) &= 7 \\(x+2)(y+2)(z+2) &= -3.\end{aligned}$$

Find, with proof, the value of $(x+3)(y+3)(z+3)$.

Solution. Let $P(t) = (t+x)(t+y)(t+z)$. Then we know t is a monic cubic such that $P(0) = -4, P(1) = 7, P(2) = -3$ and we wish to compute $P(3)$. Forming the difference table, we can immediately fill in the $6 = 3!$ in row 3 as we know P 's leading coefficient:

Row 0	-4	7	-3	?
Row 1	11	-10	?	-
Row 2	-21	?	-	-
Row 3	6	-	-	-

Then we can work our way up to get

Row 0	-4	7	-3	-28
Row 1	11	-10	-25	-
Row 2	-21	-15	-	-
Row 3	6	-	-	-

Thus, $(x+3)(y+3)(z+3) = -28$.

A more clever application of Lemma 2.2.1 yields the following result:

Theorem (Prouhet-Tarry-Escott). Let k be some natural number. There exists $e_1, e_2, \dots, e_{2^k} \in \{-1, 1\}$ such that for any polynomial $P(x)$ of degree $< k$,

$$e_1P(1) + e_2P(2) + \dots + e_{2^k}P(2^k) = 0.$$

Proof. The base case for $k = 1$ is obvious. We will show the induction step from $k - 1$ to k . Let $Q(x) = P(x) - P(x + 2^{k-1})$. By Lemma 2.2.1, $Q(x)$ is a polynomial of degree $< k - 1$. By the induction hypothesis for $k - 1$, there exists $e_1, e_2, \dots, e_{2^{k-1}}$ such that $e_1Q(1) + \dots + e_{2^{k-1}}Q(2^{k-1}) = 0$. Then

$$\begin{aligned} e_1Q(1) + \dots + e_{2^{k-1}}Q(2^{k-1}) &= 0 \\ e_1(P(1) - P(2^{k-1} + 1)) + \dots + e_{2^{k-1}}(P(2^{k-1}) - P(2^k)) &= 0 \\ e_1P(1) + \dots + e_{2^{k-1}}P(2^{k-1}) - e_1P(2^{k-1} + 1) - \dots - e_{2^k}P(2^k) &= 0. \end{aligned}$$

So this new sequence works for k as desired. By induction, we are done. □

Remark. The sequence can be defined explicitly and is known as the Thue-Morse sequence.

2.3 Lagrange Interpolation

Theorem (Lagrange Interpolation). Given n points (x_i, y_i) for $i = 1, \dots, n$ and x_i pairwise distinct, there is a unique polynomial $L(x)$ of degree $\leq n - 1$ such that $P(x_i) = y_i$ for $i = 1, \dots, n$. This polynomial can be written as

$$L(x) = \sum_{i=1}^n y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

Proof. Let us first check that the given polynomial does indeed satisfy the constraints. For any $1 \leq k \leq n$, we have

$$\begin{aligned} L(x_k) &= \sum_{i=1}^n y_i \prod_{j \neq i} \frac{x_k - x_j}{x_i - x_j} \\ &= y_k \prod_{j \neq k} \frac{x_k - x_j}{x_k - x_j} + \sum_{i \neq k} y_i \prod_{j \neq i} \frac{x_k - x_j}{x_i - x_j} \\ &= y_k \cdot 1 + \sum_{i \neq k} y_i \cdot \frac{x_k - x_k}{x_i - x_j} \cdot \prod_{j \neq i, k} \frac{x_k - x_j}{x_i - x_j} \\ &= y_k. \end{aligned}$$

So this is indeed an interpolating polynomial, and the degree is clearly $\leq n - 1$. It remains to prove that L is the unique interpolating polynomial of degree $\leq n - 1$. This is immediate by Lemma 2.1.1, so we are done. \square

Remark. If $L(x)$ is the Lagrange interpolating polynomial, then the interpolating polynomials (of any degree) are exactly $L(x) + A(x) \prod_{i=1}^n (x - x_i)$ for $A(x)$ any polynomial.

Example 3 (PFTB). Let a_1, a_2, \dots, a_n be distinct integers. Prove that for any natural number k the value $\sum_{i=1}^n \frac{a_i^k}{\prod_{j \neq i} (a_i - a_j)}$ is also an integer.

Proof. Let $P(x) = x^k$. Let $L(x)$ be the Lagrange interpolating polynomial

$$L(x) = \sum_{i=1}^n a_i^k \prod_{j \neq i} \frac{x - a_j}{a_i - a_j}.$$

The value in the problem is exactly the coefficient of x^{n-1} in $L(x)$. We know that $P(x) = L(x) + A(x) \prod_{i=1}^n (x - a_i)$ where $A(x)$ is some polynomial. Note that $A(x)$ is an integer polynomial as $P(x)$ is an integer polynomial and $\prod_{i=1}^n (x - a_i)$ is a monic integer polynomial. As $L(x) = P(x) - A(x) \prod_{i=1}^n (x - a_i) \in \mathbb{Z}[x]$, its coefficients are integers so we are done. \square

Remark. If $k < n$, the proof above tells us that the value in question is exactly 0.

2.4 Algebraic Conditions

These tend to be trickier to handle, and the method depends very much on the exact problem. Here are some general tips

- A lot of the tips for FEs apply! Things like surjectivity/injectivity aren't (usually) relevant, but you should try to
 - Figure out the solution set first
 - Try to show the polynomial is even/odd (if the solution set reflects this)
 - Plug things in to set terms to zero or cancel terms
- Extend the condition set if necessary. For example, if the problem has a condition for all x, y, z such that x, y, z are positive, you should be able to show that it's true for all x, y, z complex.
- Reduce the condition to a univariate problem if it's multivariate. This may involve treating all but one of the variables as constants and interpreting the problem as a single-variable polynomial. Then you can use the typical ideas for handling univariate polynomials.

Let's take a look at a recent USAMO P6.

Example 4 (USAMO 2019). Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$

holds for all nonzero real numbers x, y, z satisfying $2xyz = x + y + z$.

Solution. Define the multivariate polynomial

$$F(x, y, z) = xP(x) + yP(y) + zP(z) - xyz(P(x-y) + P(y-z) + P(z-x)).$$

Let d be the degree of x in F . Note that $(2yz - 1)x \equiv y + z \pmod{(2xyz - x - y - z)}$. Then

$$(2yz - 1)^d F(x, y, z) = (2xyz - x - y - z)G(x, y, z) + H(y, z)$$

for some polynomial $G(x, y, z)$ and $H(y, z)$. Then we can choose any y, z such that $y, z \neq 0$, $y + z \neq 0$, and $yz \neq \frac{1}{2}$, and compute the corresponding x such that $2xyz = x + y + z$ to obtain $H(y, z) = 0$. As H is 0 over a non-empty open set, we can conclude that H is identically 0. Furthermore, $2yz - 1$ and $2xyz - x - y - z$ are relatively prime, so we can conclude that for some $G_2(x, y, z)$,

$$F(x, y, z) = (2xyz - x - y - z)G_2(x, y, z).$$

Now, we can use any $x, y, z \in \mathbb{C}$ such that $2xyz = x + y + z$. Plugging in $(x, -x, 0)$ gives $xP(x) - xP(-x) = 0$ so P is even. Now let us plug in $(x, \frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}})$. We have

$$\begin{aligned} xP(x) + \frac{i}{\sqrt{2}}P\left(\frac{i}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}}P\left(-\frac{i}{\sqrt{2}}\right) &= \frac{x}{2} \left(P\left(x - \frac{i}{\sqrt{2}}\right) + P\left(-x - \frac{i}{\sqrt{2}}\right) + P\left(\sqrt{2}i\right) \right) \\ xP(x) &= \frac{x}{2} \left(P\left(x - \frac{i}{\sqrt{2}}\right) + P\left(x + \frac{i}{\sqrt{2}}\right) + P\left(\sqrt{2}i\right) \right) \\ P\left(x - \frac{i}{\sqrt{2}}\right) - 2P(x) + P\left(x + \frac{i}{\sqrt{2}}\right) &= -P\left(\sqrt{2}i\right). \end{aligned}$$

The LHS is a second-order finite difference, so we can conclude that $\deg P \leq 2$. Using $P(x) = ax^2 + b$ gives $P(x) = a(x^2 + 3)$ which can be checked.

3 Popular Polynomials

3.1 Sum of Powers

Theorem (Faulhaber's Formula). Let k be a non-negative integer. Prove that there is some rational polynomial $P_k(x)$ of degree $k + 1$ such that for all natural numbers n ,

$$P_k(n) = \sum_{j=1}^n j^k.$$

Proof. We will use strong induction. The base case of $k = 0$ is obvious. Assume we have proved that there exists such rational polynomial $P_j(x)$ for all $0 \leq j < k$. For any natural number r , let $Q_r(x) = \frac{x(x-1)\cdots(x-r+1)}{r!}$. Note that for natural numbers n , $Q_r(n) = \binom{n}{r}$. We have

$$Q_{k+1}(n+1) = \sum_{j=1}^n Q_k(j).$$

But we can also write $Q_k(j)$ as a degree k polynomial $Q_k(j) = c_k j^k + c_{k-1} j^{k-1} + \dots + c_0$ where the coefficients are rational. Thus,

$$\begin{aligned} Q_{k+1}(n+1) &= \sum_{j=1}^n \sum_{i=0}^k c_i j^i \\ &= \sum_{i=0}^k c_i \sum_{j=1}^n j^i \\ &= \left(\sum_{i=0}^{k-1} c_i P_j(n) \right) + c_k \sum_{j=1}^n j^i \\ \sum_{j=1}^n j^i &= \frac{Q_{k+1}(n+1)}{c_k} - \left(\sum_{i=0}^{k-1} \frac{c_i}{c_k} P_j(n) \right). \end{aligned}$$

It is easy to check that the RHS of the final line is indeed a polynomial of degree $k + 1$. So by induction, we are done. \square

Remark. Another similar proof by induction considers the sum $\sum_{j=1}^n j^{k+1} - (j-1)^{k+1}$.

In other words, the sum of a polynomial at spaced intervals creates a new polynomial with degree one higher.

Lemma 3.1.1. For any polynomial $Q(x)$ of degree k and constants a, b , there is some polynomial P of degree $k + 1$ such that

$$P(n) = \sum_{j=1}^n Q(aj + b).$$

Remark. In a sense, this is the reverse of finite differences. Instead of lowering the degree by one, it raises it.

3.2 Chebyshev Polynomials

Theorem (Chebyshev Polynomials). For $n \in \mathbb{N}$, there is a unique polynomial $T_n(x)$ such that $T_n(\cos \theta) = \cos(n\theta)$ for any θ . Furthermore, $T_n(x)$ is a degree n integer polynomial with leading coefficient 2^{n-1} .

Proof. By the trigonometric identity $\cos(n\theta) + \cos((n-2)\theta) = 2\cos\theta\cos((n-1)\theta)$, we have

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),$$

and $T_0(x) = 1, T_1(x) = x$. The fact that $T_n(x)$ is a degree n integer polynomial with leading coefficient 2^{n-1} follows in a straightforward manner from this recurrence. \square

Remark. The roots of the Chebyshev polynomials, $\cos\left(\frac{2k-1}{2n}\pi\right)$, are good guesses for solution sets in $[0, 1]^n$.

4 The Geometry of Polynomials

Polynomials are very well-behaved, being both continuous and differentiable. These properties allow us to deduce information about the roots of a polynomial even if we do not explicitly know their locations.

4.1 Continuity

Theorem (Intermediate Value Theorem). If real-valued function f is continuous over $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the interval.

Lemma 4.1.1. If $P(x)$ is a polynomial with real coefficients and has odd degree, it has at least one real root.

Proof. Without loss of generality, consider $P(x)$ monic. As $x \rightarrow \infty, P(x) \rightarrow \infty$ and as $x \rightarrow -\infty, P(x) \rightarrow -\infty$. So $P(x)$ must be 0 for some x . \square

Remark. This also follows from Lemma 1.0.1.

Example 5 (Minimality of Chebyshev Polynomials). Let $P(x)$ be a monic polynomial of degree $\leq n$. Then $\max_{x \in [-1, 1]} P(x) \geq \frac{1}{2^{n-1}}$, with equality when $P(x) = \frac{T_n(x)}{2^{n-1}}$.

Proof. Assume for the sake of contradiction that we have monic polynomial $P(x)$ of degree n such that $\max_{x \in [-1, 1]} P(x) < \frac{1}{2^{n-1}}$. Then define $Q(x) = \frac{T_n(x)}{2^{n-1}} - P(x)$. Note that $Q(x)$ has degree $\leq n-1$. For $x = \cos \frac{2k\pi}{n}, 0 \leq 2k \leq n$, we have

$$Q(x) = \frac{1}{2^{n-1}} - P(x) > 0.$$

For $x = \cos \frac{(2k+1)\pi}{n}$, $0 \leq 2k+1 \leq n$, we have

$$Q(x) = -\frac{1}{2^{n-1}} - P(x) < 0.$$

By the Intermediate Value Theorem, Q must have n roots. But $Q(x)$ has degree $\leq n-1$ and $Q(x) \neq 0$, so contradiction. \square

Remark. In fact, equality holds iff $P(x) = \frac{T_n(x)}{2^{n-1}}$.

4.2 Derivatives

In the previous example, it worked out well that $Q(x)$ was either strictly positive or negative at the locations we chose. What if we have a polynomial P such that $P(-1) = 1$, $P(0) = 0$, and $P(1) = 1$. What can we say about the number of roots (counting multiplicity) that P has in $(-1, 1)$? Edge cases like these can be handled by using the derivative. In particular, the derivative is effective at analyzing repeated roots.

Definition. For a differentiable function $f(x)$, its derivative $f'(x)$ denotes the slope of the tangent to f at x .

Remark. The optima of a differentiable function correspond to the zeros of its derivative (but its derivative may have zeros that are not optima).

Lemma 4.2.1 (Derivative Rules). We have the following rules for taking derivatives:

- Constant Rule: The derivative of a constant is 0.
- Product by a Constant Rule: $(cf(x))' = cf'(x)$
- Sum Rule: $(f(x) + g(x))' = f'(x) + g'(x)$
- Product Rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- Power Rule: $(x^n)' = nx^{n-1}$
- Chain Rule: $(f(g(x)))' = f'(g(x))g'(x)$

We won't go over any rigorous proof or background for these, but it may be helpful to generally know where they come from. Without going into details, the recipe for derivatives is to consider $\frac{f(x+h)-f(x)}{h}$ and look for the terms independent of h . The handwavy justification is that h is infinitesimally small and so any terms with h go to 0. For example, let's look at the power rule when n is a natural number. By the binomial formula, we have

$$\begin{aligned} \frac{(x+h)^n - x^n}{h} &= \frac{(\sum_{i=0}^n \binom{n}{i} x^{n-i} h^i) - x^n}{h} \\ &= \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + h^n}{h} \\ &= nx^{n-1} + \binom{n}{2}x^{n-2}h + \dots + h^{n-1} \\ &= nx^{n-1} + O(h). \end{aligned}$$

Example 6. What is the derivative of $(x^4 + 3x + 1)^2$?

Proof. Let $f(x) = x^2$ and $g(x) = x^4 + 3x + 1$. The power rule tells us that $f'(x) = 2x$ and $g'(x) = 4x^3 + 3$. So by the chain rule, we have

$$(f(g(x)))' = f'(g(x))g'(x) = 2(x^4 + 3x + 1)(4x^3 + 3).$$

Alternatively, we could have expanded $(x^4 + 3x + 1)^2$ to individual powers and applied the power rule to obtain the same result, although it would not be factorized for free. \square

Lemma 4.2.2 (Derivative of a Polynomial). Let non-constant polynomial $P(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_0$ have roots r_1, r_2, \dots, r_n . Then we have two formulas for its derivative:

$$P'(x) = \sum_{i=0}^{n-1} (i+1)c_{i+1}x^i$$

and

$$P'(x) = c_n \sum_{i=1}^n \prod_{j \neq i} (x - r_j) = P(x) \left(\sum_{i=1}^n \frac{1}{x - r_i} \right).$$

Note in particular that the degree of $P'(x)$ is always one less than the degree of $P(x)$ for P non-constant.

Lemma 4.2.3 (Repeated Roots). Let polynomial $P(x)$ have root r with multiplicity $k \geq 1$. Then r is a root of $P'(x)$ with multiplicity $k - 1$ (or if $k = 1$, r is not a root of $P'(x)$).

Proof. As $P(x)$ has root r with multiplicity k , we can write $P(x) = (x-r)^k Q(x)$ for some polynomial $Q(x)$ which does not have r as a root. Then by the product rule,

$$P'(x) = k(x-r)^{k-1}Q(x) + (x-r)^k Q'(x) = (x-r)^{k-1} (kQ(x) + (x-r)Q'(x)).$$

Note that plugging in $x = r$ to $kQ(x) + (x-r)Q'(x)$ gives $kQ(r) \neq 0$, so we see that the multiplicity of r in $P'(x)$ is exactly $k - 1$. \square

The derivative of a polynomial can be used to analyze a polynomial's repeated roots using the above lemma. We will see how this is done in the following example.

Example 7 (Putnam 1956). The nonconstant polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z) + 1$ and $Q(z) + 1$. Prove that $P(z) = Q(z)$.

Proof. Assume for the sake of contradiction that $P \neq Q$. Let P have degree n and Q have degree m . Without loss of generality, $n \geq m$. Assume that $P(z)$ has j distinct roots and $P(z) + 1$ has k distinct roots. Note that the set of roots of $P(z)$ is disjoint with the set of roots of $P(z) + 1$. Since

$$P(z) - Q(z) = (P(z) + 1) - (Q(z) + 1),$$

note that $P(z) - Q(z)$ has at least $j + k$ roots. $P(z) - Q(z)$ is not the zero polynomial and has degree $\leq n$, so $j + k \leq n$. However, consider $P'(z) = (P(z) + 1)'$. Lemma 4.2.3 tells us that $P'(z)$ shares at least $n - j$ roots (counted with multiplicity) with $P(z)$. It also tells us that $P'(z)$ shares at least $n - k$ roots with $P(z) + 1$. Again, these two sets are disjoint. So $P'(z)$ has at least $2n - j - k$ roots. But $\deg P' = n - 1$, so $n - 1 \geq 2n - j - k \implies j + k \geq n + 1$. So there is a contradiction. \square

Theorem (Gauss-Lucas). If P is a non-constant polynomial with complex coefficients, all zeroes of P' belong to the convex hull of the set of zeros of P .

Proof. Let us write $P(x) = c \prod_{i=1}^n (x - r_i)$. From Lemma 4.2.2, we have $\frac{P'(x)}{P(x)} = \sum_{i=1}^n \frac{1}{x - r_i}$. Let z be a root of P' . If z is also a root of P , it is trivially within the convex hull. Otherwise, we have

$$\begin{aligned} 0 &= \frac{P'(z)}{P(z)} \\ &= \sum_{i=1}^n \frac{1}{z - r_i} \\ &= \sum_{i=1}^n \frac{\bar{z} - \bar{r}_i}{|z - r_i|^2} \\ \sum_{i=1}^n \frac{z}{|z - r_i|^2} &= \sum_{i=1}^n \frac{r_i}{|z - r_i|^2} \\ z &= \sum_{i=1}^n r_i \frac{|z - r_i|^{-2}}{\sum_{j=1}^n |z - r_j|^{-2}}. \end{aligned}$$

So z can indeed be written as a convex combination of the roots of P . \square

Example 8 (Putnam 2018). Let n be a positive integer, and let $f_n(z) = n + (n - 1)z + (n - 2)z^2 + \dots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

Proof. Let $g_n(z) = z^n + z^{n-1} + \dots + 1$. Note that $f_n(z)$ strongly resembles $g'_n(z)$ except the coefficients appear to be reversed. We can handle this reverse by considering $f_n(\frac{1}{z})$. In particular, we have

$$g'_n(z) = nz^{n-1} + (n-1)z^{n-2} + \dots + 1 = z^{n-1} f_n\left(\frac{1}{z}\right).$$

Note that $z = 0$ is clearly not a root of f_n . So for $z \neq 0$, $\frac{1}{z}$ is a root of f_n iff z is a root of g'_n . By Gauss-Lucas, the roots of g'_n are contained in the convex hull of the roots of g_n . But observe that the roots of g_n are roots of unity, and so lie on the closed unit disk. Furthermore, g_n has no double-roots, so g'_n does not share any roots. Thus, all the roots of g'_n lie in the open unit disk, and so all the roots of f_n lie outside the closed unit disk. \square

Remark. This problem can also be solved without using Gauss-Lucas, but the above gives a deeper reason for why it's true.

4.3 Interlacing

The location of roots of P also gives information on the location of roots of P' . This is called interlacing, as the graphs of P and P' appear to be interlaced. To end off this handout, we will use interlacing to prove Descartes' Rule of Signs.

Theorem (Rolle's Theorem). If real-valued function f is continuous over $[a, b]$, differentiable over (a, b) , and $f(a) = f(b)$, then there exists at least one c in (a, b) such that $f'(c) = 0$.

From Rolle's Theorem and Lemma 4.2.3, the idea of interlacing follows.

Lemma 4.3.1 (Interlacing). If $f(x)$ is a differentiable function in interval (a, b) and $f(x) = 0$ has k roots (counting multiplicity) in (a, b) then $f'(x) = 0$ has at least $k - 1$ roots in (a, b) .

Theorem (Descartes' Rule of Signs). Consider polynomial $P(x) = \sum_{i=1}^k c_i x^{p_i}$ with $0 \leq p_1 < p_2 + \dots + p_k$ and non-zero real coefficients c_i . Let $V(P)$ be the number of sign changes in the coefficients of P , i.e. where $c_i c_{i+1} < 0$. Let $Z(P)$ be the number of strictly positive roots of $P(x)$. Then $Z(P) \leq V(P)$ and $V(P) - Z(P)$ is even.

Proof. We only need to consider $p_1 = 0$ as dividing by x^{p_1} does not change $Z(P)$ or $V(P)$. Let us first prove that $Z(P)$ and $V(P)$ share the same parity. Note that $P(0) = c_0$ and $\text{sgn}(P(\infty)) = \text{sgn}(c_k)$. If $c_0 > 0$ and $c_k < 0$ for example, there are an odd number of positive roots and there an odd number of sign changes from c_0 to c_k . Other cases are similar.

Now we will show that $Z(P) \leq V(P)$ by inducting over the degree. The base case is true for constant polynomials. Assume that Descartes' Rule of Signs is true for polynomials of degree $\leq \deg(P) - 1$. As $\deg(P') \leq \deg(P) - 1$, we can assume the inductive hypothesis with P' . This tells us that $V(P') = Z(P') + 2s$ for some non-negative integer s . By Lemma 4.3.1 on interval $(0, \infty)$, we have $Z(P') \geq Z(P) - 1$. Additionally, we have that

$$P'(x) = \sum_{i=1}^k c_i p_i x^{p_i-1},$$

so $V(P')$ will either remain the same or be one less than $V(P)$. Thus, we have

$$Z(P) \leq Z(P') + 1 \leq V(P') - 2s + 1 \leq V(P) - 2s + 1 \leq V(P) + 1.$$

As $Z(P)$ and $V(P)$ are the same parity, we can conclude that $Z(P) \leq V(P)$. So we are done by induction. \square

A Problems

A1. (*HMMT 2017*) Let $P(x)$, $Q(x)$ be nonconstant polynomials with real coefficients. Prove that if

$$\lfloor P(y) \rfloor = \lfloor Q(y) \rfloor$$

for all real numbers y , then $P(x) = Q(x)$ for all real numbers x .

A2. (*CMO 2013*) Determine all polynomials $P(x)$ with real coefficients such that

$$(x+1)P(x-1) - (x-1)P(x)$$

is a constant polynomial.

A3. (*Spain 2002*) Find all polynomials $P(t)$ of one variable that fulfill the following for all $x, y \in \mathbb{R}$:

$$P(x^2 - y^2) = P(x - y)P(x + y).$$

A4. (*USAMO 2014*) Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3 , and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

A5. (*Unknown*) A polynomial p of degree n satisfies $p(k) = 2^k$ for all $0 \leq k \leq n$. Find $p(n+1)$.

A6. (*Putnam 2023*) Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \dots + a_1x + a_0$ for some real coefficients a_0, \dots, a_{2n-1} . Suppose that $p(1/k) = k^2$ for all integers k such that $1 \leq |k| \leq n$. Find all other real numbers x for which $p(1/x) = x^2$.

A7. (*APMO 2009*) Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying the following equations:

$$\frac{a_1}{k^2 + 1} + \frac{a_2}{k^2 + 2} + \frac{a_3}{k^2 + 3} + \frac{a_4}{k^2 + 4} + \frac{a_5}{k^2 + 5} = \frac{1}{k^2}$$

for $k = 1, 2, 3, 4, 5$. Find the value of $\frac{a_1}{37} + \frac{a_2}{38} + \frac{a_3}{39} + \frac{a_4}{40} + \frac{a_5}{41}$.

A8. (*Serbia 2013*) Let $P(x)$ be a polynomial of degree 2012 with real coefficients satisfying the condition

$$P(a)^3 + P(b)^3 + P(c)^3 \geq 3P(a)P(b)P(c),$$

for all real numbers a, b, c such that $a + b + c = 0$. Is it possible for $P(x)$ to have exactly 2012 distinct real roots?

A9. (*COMC 2014*) A polynomial $f(x)$ with real coefficients is said to be a *sum of squares* if there are polynomials $p_1(x), p_2(x), \dots, p_n(x)$ with real coefficients for which

$$f(x) = p_1^2(x) + p_2^2(x) + \dots + p_n^2(x).$$

Suppose $f(x)$ is a sum of squares. Prove there are polynomials $u(x), v(x)$ with real coefficients such that $f(x) = u^2(x) + v^2(x)$.

A10. (*Putnam 2014*) Show that for each positive integer n , all the roots of the polynomial

$$\sum_{k=0}^n 2^{k(n-k)} x^k$$

are real numbers.

A11. (*Unknown*) Let $P(x)$ be a polynomial of degree n with real coefficients such that $|f(x)| \leq 1$ for all $x \in [0, 1]$. Show that $|f(-\frac{1}{n})| \leq 2^{n+1} - 1$.

A12. (*Serbia 2023*) Let p be a prime and $P \in \mathbb{R}[x]$ be a polynomial of degree less than $p - 1$ such that $|P(1)| = |P(2)| = \dots = |P(p)|$. Prove that P is constant.

A13. (*Putnam 2019*) Given real numbers $b_0, b_1, \dots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \dots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{2019} b_k z^k.$$

Let $\mu = (|z_1| + \dots + |z_{2019}|)/2019$ be the average of the distances from $z_1, z_2, \dots, z_{2019}$ to the origin. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \dots, b_{2019}$ that satisfy

$$1 \leq b_0 < b_1 < b_2 < \dots < b_{2019} \leq 2019.$$

.

B Problems

B1. (*Serbia 2018*) For any positive integer n , define

$$c_n = \min_{(z_1, z_2, \dots, z_n) \in \{-1, 1\}^n} |z_1 \cdot 1^{2018} + z_2 \cdot 2^{2018} + \dots + z_n \cdot n^{2018}|.$$

Is the sequence $(c_n)_{n \in \mathbb{Z}^+}$ bounded?

B2. (*CMO 2018*) Find all polynomials $p(x)$ with real coefficients that have the following property: There exists a polynomial $q(x)$ with real coefficients such that

$$p(1) + p(2) + p(3) + \dots + p(n) = p(n)q(n)$$

for all positive integers n .

B3. (*Olympic Revenge 2020*) Let n be a positive integer and a_1, a_2, \dots, a_n non-zero real numbers. What is the least number of non-zero coefficients that the polynomial $P(x) = (x - a_1)(x - a_2) \cdots (x - a_n)$ can have?

B4. (*Kürschák 2016*) If $p, q \in \mathbb{R}[x]$ satisfy $p(p(x)) = q(x)^2$, does it follow that $p(x) = r(x)^2$ for some $r \in \mathbb{R}[x]$?

B5. (*ISL 2019*) Let x_1, x_2, \dots, x_n be different real numbers. Prove that

$$\sum_{1 \leq i \leq n} \prod_{j \neq i} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

B6. (*China 1995*) A and B play the following game with a polynomial of degree at least 4:

$$x^{2n} + \square x^{2n-1} + \square x^{2n-2} + \dots + \square x + 1 = 0$$

A and B take turns to fill in one of the blanks with a real number until all the blanks are filled up. If the resulting polynomial has no real roots, A wins. Otherwise, B wins. If A begins, which player has a winning strategy?

B7. (*RMM 2018*) Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P(x)^{10} + P(x)^9 = Q(x)^{21} + Q(x)^{20}.$$

B8. (*ELMOSL 2012*) Prove that any polynomial of the form $1 + a_n x^n + a_{n+1} x^{n+1} + \cdots + a_k x^k$ ($k \geq n$) has at least $n - 2$ non-real roots (counting multiplicity), where the a_i ($n \leq i \leq k$) are real and $a_k \neq 0$.

B9. (*IMO 2016*) The equation

$$(x - 1)(x - 2) \cdots (x - 2016) = (x - 1)(x - 2) \cdots (x - 2016)$$

is written on the board, with 2016 linear factors on each side. What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

B10. (*ISL 2020*) A magician intends to perform the following trick. She announces a positive integer n , along with $2n$ real numbers $x_1 < \cdots < x_{2n}$, to the audience. A member of the audience then secretly chooses a polynomial $P(x)$ of degree n with real coefficients, computes the $2n$ values $P(x_1), \dots, P(x_{2n})$, and writes down these $2n$ values on the blackboard in non-decreasing order. After that the magician announces the secret polynomial to the audience. Can the magician find a strategy to perform such a trick?

B11. (*Russia 1997*) Let a_1, a_2, \dots, a_m be nonzero real numbers satisfying

$$1^k \cdot a_1 + 2^k \cdot a_2 + \cdots + m^k \cdot a_m = 0$$

for each $k = 0, 1, \dots, n$. Prove that the sequence a_1, a_2, \dots, a_m changes sign at least $n + 1$ times.

C Problems

C1. (*USAMO 2002*) Prove that any monic polynomial of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

C2. (*Kürschák 2017*) Do there exist polynomials $p(x)$ and $q(x)$ with real coefficients such that $p^3(x) - q^2(x)$ is linear but not constant?

C3. (*ISL 2019*) A polynomial $P(x, y, z)$ in three variables with real coefficients satisfies the identities

$$P(x, y, z) = P(x, y, xy - z) = P(x, zx - y, z) = P(yz - x, y, z).$$

Prove that there exists a polynomial $F(t)$ in one variable such that

$$P(x, y, z) = F(x^2 + y^2 + z^2 - xyz).$$

C4. (*USATSTST 2017*) Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)},$$

where f and g are polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f , or show that no such f, g exist.

C5. (*ISL 2014*) Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers x and y one has

$$|y^2 - P(x)| \leq 2|x| \quad \text{if and only if} \quad |x^2 - P(y)| \leq 2|y|.$$

Determine all possible values of $P(0)$.

C6. (*ISL 2013*) Let $m \neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

$$(x^3 - mx^2 + 1)P(x + 1) + (x^3 + mx^2 + 1)P(x - 1) = 2(x^3 - mx + 1)P(x)$$

for all real number x .

C7. (*Putnam 2022*) Let n be a positive integer. Determine, in terms of n , the largest integer m with the following property: There exist real numbers x_1, \dots, x_{2n} with $-1 < x_1 < x_2 < \dots < x_{2n} < 1$ such that the sum of the lengths of the n intervals

$$[x_1^{2k-1}, x_2^{2k-1}], [x_3^{2k-1}, x_4^{2k-1}], \dots, [x_{2n-1}^{2k-1}, x_{2n}^{2k-1}]$$

is equal to 1 for all integers k with $1 \leq k \leq m$.

C8. (*USATST 2017*) Let $P, Q \in \mathbb{R}[x]$ be relatively prime nonconstant polynomials. Show that there can be at most three real numbers λ such that $P + \lambda Q$ is the square of a polynomial.

C9. (*ELMOSL 2011*) Let $n > 1$ be an integer and a, b, c be three complex numbers such that $a + b + c = 0$ and $a^n + b^n + c^n = 0$. Prove that two of a, b, c have the same magnitude.

C10. (*China 2018*) Suppose the real number $\lambda \in (0, 1)$, and let n be a positive integer. Prove that the modulus of all the roots of the polynomial

$$f(x) = \sum_{k=0}^n \binom{n}{k} \lambda^{k(n-k)} x^k$$

are 1.

C11. (*ISL 2015*) Let n be a fixed integer with $n \geq 2$. We say that two polynomials P and Q with real coefficients are block-similar if for each $i \in \{1, 2, \dots, n\}$ the sequences

$$P(2015i), P(2015i - 1), \dots, P(2015i - 2014) \quad \text{and} \\ Q(2015i), Q(2015i - 1), \dots, Q(2015i - 2014)$$

are permutations of each other.

(a) Prove that there exist distinct block-similar polynomials of degree $n + 1$.

(b) Prove that there do not exist distinct block-similar polynomials of degree n .