Tangent Circles

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Problems concerning tangent circles are challenging, but — perhaps due to this difficulty — the solutions often end up similar to each other. There are three synthetic approaches which are very effective:

- **Inversion**
  Inversion is always worth trying. In the case of tangent circles, one common trick is to invert one of the circles into a line while fixing the other and doing an overlay.
- **Locating the point of tangency**
  Often, the point of tangency can be described in a different way. I've found that 80% of the time, it's the Miquel point of some triangle or complete quad in the diagram. If you manage to identify the point, then you can just follow this easy procedure:
  1. Redefine point T as whatever you think it might be.
  2. Show that it's on each circle (probably just angle-chasing).
  3. Hopefully you can angle-chase the rest.
- **Forming a homothety**
  Tangent circles are homothetic with their point of tangency as the center. A solution path is to find/construct triangles on each of the circles which are homothetic.

Of course, there are other approaches, a few of which are:

- **Bashing**
  Have fun.
- **Casey's Theorem**
  See here for the statement. In particular, say you have three points X, Y, Z and a circle Γ. Let t_X be the length of the tangent from X to Γ and define t_Y and t_Z similarly. If you prove that
  $$t_X \cdot YZ + t_Y \cdot ZX = t_Z \cdot XY$$
  or something cyclic to that (same order as Ptolemy's), then the converse of Casey's implies that Γ is tangent to the circumcircle of $\triangle XYZ$.
- **Curvilinear incircles**
  Knowing certain configurations (Sawayama's Theorem, mixtilinear incircles, etc.) is helpful and can even trivialize some problems. See Yufei Zhao's notes: 1 and 2.

A Problems

A1. (INMO 2019) Let AB be the diameter of a circle Γ and let C be a point on Γ different from A and B. Let D be the foot of perpendicular from C on to AB. Let K be a point on the segment CD such that AC is equal to the semi perimeter of ADK. Show that the excircle of ADK opposite A is tangent to Γ.

A2. Let ABC be a triangle and let B' and C' be points on AB and AC such that $B'C' \parallel BC$. Show that there exists a circle passing through B' and C' that is tangent to the incircle and A-excircle of ABC.

A3. (RMM 2018). Let $ABCD$ be a cyclic quadrilateral and let P be a point on the side AB. The diagonal AC meets DP at Q. The line through P parallel to CD meets the extension of the side CB beyond B at K. The line through Q parallel to BD meets the extension of the side CB beyond B at L. Prove that the circumcircles of $\triangle BKP$ and $\triangle CLQ$ are tangent.

A4. (Iran 2013). Let $ABCDE$ be a pentagon inscribe in a circle (O). Let $BE \cap AD = T$. Suppose the parallel line with CD which passes through T which cut AB, CE at X, Y. If ω be the circumcircle of triangle AXY then prove that ω is tangent to (O).

A5. Let $ABCD$ be a rectangle. Suppose that Γ is a circle which passes through A and C (but not all four points). Two circles ω_1 and ω_2 lie within $ABCD$ such that ω_1 is tangent to BA, BC, and Γ, and ω_2 is tangent to DA, DC, and Γ. Prove that the sum of the radii of ω_1 and ω_2 is independent of the choice of Γ.

A6. (APMO 2006). Let A, B be two distinct points on a given circle O and let P be the midpoint of the line segment AB. Let O_1 be the circle tangent to the line AB at P and tangent to the circle O. Let ℓ be the tangent line, different from the line AB, to O_1 passing through A. Let C be the intersection point, different from A, of ℓ and O. Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC. Prove that the circle O_2 is tangent to the circle O.

A7. (ISL 2018). Let O and Ω be the circumcenter and circumcircle respectively of acute triangle ABC. Let P be an arbitrary point on Ω, distinct from A, B, C, and their antipodes in Ω. Denote the circumcenters of the triangles AOP, BOP, and COP as O_A, O_B, and O_C, respectively. The lines ℓ_A, ℓ_B, ℓ_C perpendicular to BC, CA, and AB pass through O_A, O_B, and O_C, respectively. Prove that the circumcircle of triangle formed by ℓ_A, ℓ_B, and ℓ_C is tangent to the line OP.

B Problems

B1. (ISL 2017). In $\triangle ABC$, let ω be the excircle opposite to A. Let D, E, and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circumcircle of $\triangle MPQ$ is tangent to ω.

B2. (CMO 2015). Let $\triangle ABC$ be an acute triangle with circumcenter O. Let I be a circle with center on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that $BP \cdot CQ = AP \cdot AQ$. Prove that I is tangent to the circumcircle of $\triangle BOC$.

B3. (RMM 2016). A hexagon convex $A_1B_1A_2B_2A_3B_3$ it is inscribed in a circumference Ω with radius R. The diagonals A_1B_2, A_2B_3, A_3B_1 are concurrent in X. For each $i = 1, 2, 3$ let ω_i tangent to the segments XA_i and XB_i and tangent to the arc A_iB_i of Ω that does not contain the other vertices of the hexagon; let r_i the radius of ω_i.

a) Prove that $R \geq r_1 + r_2 + r_3$.

b) If $R = r_1 + r_2 + r_3$, prove that the six points of tangency of the circumferences ω_i with the diagonals A_1B_2, A_2B_3, A_3B_1 are concyclic.

B4. (Iran 2017). In triangle ABC, points P and Q lie on side BC such that $BP = CQ$ and P lies between B and Q. The circumcircle of $\triangle APQ$ intersects sides AB and AC at E and F, respectively. The point T is the intersection of EP and FQ. Two lines passing through the midpoint of BC and parallel to AB and AC intersect EP and FQ at points X and Y, respectively. Prove that the circumcircles of $\triangle TXY$ and $\triangle APQ$ are tangent to each other.

B5. (ISL 2002). The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

B6. (ISL 2018). Let ABC be a triangle with circumcircle Ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to Ω.

B7. (Taiwan 2019). Given $\triangle ABC$, denote its incenter and orthocenter by I and H, respectively. Assume there is a point K with

$$AH + AK = BH + BK = CH + CK.$$

Show that K lies on line HI.

C Problems

C1. (APMO 2014). Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q, and AB is tangent to Ω.

C2. (ELMO 2016). Elmo is now learning olympiad geometry. In $\triangle ABC$ with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\angle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\angle XSY = \angle XTY = 90°$. Finally, let γ be the circumcircle of $\triangle AST$.

a) Help Elmo show that γ is tangent to the circumcircle of $\triangle ABC$.

b) Help Elmo show that γ is tangent to the incircle of $\triangle ABC$.

C3. (RMM 2013). Let $ABCD$ be a quadrilateral inscribed in a circle ω. The lines AB and CD meet at P, the lines AD and BC meet at Q, and the diagonals AC and BD meet at R. Let M be the midpoint of the segment PQ and let K be the common point of the segment MR and the circle ω. Prove that the circumcircle of $\triangle KPQ$ and ω are tangent.

C4. (Iran 2012). Suppose $ABCD$ is a parallelogram. Consider circles ω_1 and ω_2 such that ω_1 is tangent to segments AB and AD and ω_2 is tangent to segments BC and CD. Suppose that there exists a circle which is tangent to lines AD and DC and externally tangent to ω_1 and ω_2. Prove that there exists a circle which is tangent to lines AB and BC and also externally tangent to circles ω_1 and ω_2.

C5. (IMO 2011). Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b, and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, and ℓ_c is tangent to the circle Γ.

C6. (RMM 2018). Fix a circle Γ, a line ℓ to tangent Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω meet ℓ at Y and Z. Prove that, as X varies over Ω, the circumcircle of XYZ is tangent to two fixed circles.