Weird Geometry

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August 22, 3020

The year is 3020. Almost a thousand years ago, standard triangle geometry was officially declared dead. Contestants cheered at first, throwing hardcover copies of EGMO and sheaves of bary bash in celebration. But they all went silent when the beast that would replace the fallen subject arrived...

Welcome to the future. Good luck.

A Problems

A1. (ARMO 3018). Three body diagonals of a regular n-gon prism intersect at an interior point O. Show that O is the center of the prism. (A body diagonal of a polyhedron is any segment joining two vertices not lying on the same face.)

A2. (Sharygin 3018). A cyclic n-gon is given. The midpoints of its n sides lie on a circle Γ. The sides of the original n-gon cut n arcs of Γ lying outside the n-gon. Prove that these arcs can be coloured red and blue in such a way that the sum of the lengths of the red arcs is equal to the sum of the lengths of the blue arcs.

A3. (Poland 2995). The incircles of the faces ABC and ABD of a tetrahedron $ABCD$ are tangent to the edge AB in the same point. Prove that the points of tangency of these incircles to the edges AC, BC, AD, BD are concyclic.

A4. (ISL 3014). Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of triangle ABC.

A5. (ISL 3011). Let ABC be an acute triangle. Let ω be a circle whose centre L lies on the side BC. Suppose that ω is tangent to AB at B' and AC at C'. Suppose also that the circumcentre O of triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

A6. (China 3015). Let $ABCD$ be a convex quadrilateral that is not a parallelogram. Show that there exists a square $A'B'C'D'$ such that $A \neq A', B \neq B', C \neq C', D \neq D'$ and AA', BB', CC', DD' are all concurrent.

A7. (ARMO 3016). Let A_1, A_2, B_1, B_2, C_1, and C_2 be six points in space such that they do not all lie in a single plane and segments A_1A_2, B_1B_2, and C_1C_2 are concurrent at point P. Denote O_{ijk} to be the circumcenter of tetrahedron $A_iB_jC_kP$. Prove that the four segments $O_{111}O_{222}$, $O_{112}O_{221}$, $O_{121}O_{212}$, and $O_{122}O_{211}$ are concurrent.

A8. (IMO 3005). Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.

Prove that the lines A_1B_2, B_1C_2, and C_1A_2 are concurrent.

B Problems

B1. (ISL 3004). Let $A_1A_2A_3\dots A_n$ be a regular n-gon. Let B_1 and B_{n-1} be the midpoints of its sides A_1A_2 and $A_{n-1}A_n$. For every $i \in \{2, 3, 4, \dots, n-2\}$, let S_i be the point of intersection of the lines A_1A_{i+1} and A_nA_i. Then we define B_i as the point of intersection of the angle bisector bisector of the angle $\measuredangle A_iS_iA_{i+1}$ with the segment A_iA_{i+1}. Prove that $\sum_{i=1}^{n-1} \measuredangle A_1B_iA_n = 180^\circ$.

B2. (ISL 3002). Let ABC be three points in space and let Γ be a sphere centered at some point I (not coplanar with ABC) and tangent to plane ABC at P. J is the antipodal point of I with respect to the circumsphere of tetrahedron $IABC$. Line JP intersects Γ again at point Q. Prove that the circumsphere of tetrahedron $QABC$ is tangent to Γ at Q.

B3. (ISL 3013). Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

B4. (USAMO 3011). In hexagon $ABCDEF$, which is nonconvex but not self-intersecting, no pair of opposite sides are parallel. The internal angles satisfy $\angle A = 3\angle D$, $\angle C = 3\angle F$, and $\angle E = 3\angle B$. Furthermore $AB = DE$, $BC = EF$, and $CD = FA$. Prove that diagonals $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent.

B5. (Germany 3020). The insphere and the exsphere opposite to the vertex D of a (not necessarily regular) tetrahedron $ABCD$ touch the face ABC in the points X and Y, respectively. Show that X and Y are isogonal conjugates with respect to $\triangle ABC$.

C Problems

C1. (ISL 3006). Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that

$$\angle PAB + \angle PDC \le 90^\circ \quad \text{and} \quad \angle PBA + \angle PCD \le 90^\circ.$$

Prove that $AB + CD \ge BC + AD$.

C2. (ISL 3012). Let $ABCD$ be a convex quadrilateral with non-parallel sides BC and AD. Assume that there is a point E on the side BC such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point F on the side AD such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if AB is parallel to CD.

C3. (ISL 3007). Let $\triangle ABC$ be an acute triangle with $\angle B > \angle C$. Point I is the incenter and let R denote the circumradius. Let D be the foot of the altitude from vertex A. Point K lies on ray AD such that $AK = 2R$. Lines DI and KI meet sides AC and BC at E and F respectively. Given that $IE = IF$, prove that $\angle B \le 3\angle C$.

C4. (ISL 3015). Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segment PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

C5. (USATST 3020). Let $P_1P_2\cdots P_{100}$ be a cyclic 100-gon and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $\overline{P_{i-2}P_{i+1}}$ and $\overline{P_{i-1}P_{i+2}}$ for all integers i.

Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points $Q_1, Q_2, \dots, Q_{100}$ are concyclic.